^{3}He-^{4}He liquid mixtures and spin-polarized ^{3}He↑ quantum systems

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CONTENTS:

5. **Dense ^{3}He$\uparrow$ liquid systems.**
    - **5.1** Methods of polarization.
    - **5.2** Spin-polarized Fermi liquids.
    - **5.3** Spin dynamics and spin waves.
6. **Conclusions.**
   Experimental possibilities

LITERATURE (reviews)

- E.P. Bashkin, A.E. Meyerovich
  ^{3}He-^{4}He quantum solutions.
  Adv. Phys. 30 (1981) 1-92
    - Relatively old but detailed review on ^{3}He-^{4}He mixtures.
- A.E. Meyerovich
  Spin-polarized ^{3}He-^{4}He solutions.
  Progr. Low Temp. Phys. 11 (1987) 1-72
    - A much more recent review on polarized mixtures (but not very detailed).
- A.E. Meyerovich
  Spin-polarized phases of ^{3}He.
  in: Helium three, eds. W. Halperin, L. Pitaevski, North-Holland- Amsterdam 1990, p.757-880
    - The most detailed and recent review (covers the literature up to 1989).

1 Possible approaches to ^{3}He-^{4}He mixtures

1.1 ^{3}He impurity system in a superfluid ^{4}He: nearly an ideal Fermi gas

Main parameters of the system

In this course, we will mostly consider liquid solutions of ^{3}He atoms in superfluid ^{4}He (He II). The main parameters of the system are:

1. ^{3}**He concentration:**

   $$x = N_3/(N_3 + N_4)$$

   where $N_{3,4}$ $[cm^{-3}]$ are the numbers of ^{3}He (^{4}He) atoms per unit volume of the system.

   $$x \sim A_0^3 N_3$$

   where $A_0 \sim 3 \times 10^{-8}$ cm is the atomic size.
   x is the only small parameter of the system. At zero pressure (rather, at SVP-saturated vapour pressure) $x \leq 0.065$. At the concentration $x > 6.5\%$, the solution demixes (at low temperatures) into pure ^{3}He and ^{3}He-^{4}He mixture with $x = 6.5\%$. At higher pressures, the limiting concentration increases to about 9%.

2. **Temperature.** We will assume that the temperature T in most of the cases is less than $T < 0.2$ K so that one can nearly always neglect the presence of thermal phonons and rotons in the system.

3. **Degeneracy (Fermi) temperature** for the ^{3}He component.

   $$T_F \approx \frac{(3\pi^2)^{\frac{2}{3}}}{2} \frac{\hbar^2}{m^*} \sim (at\ SVP) \sim 2.6 x^{\frac{2}{3}} [K] \tag{1.1}$$

   where m^* is the effective mass of ^{3}He quasi-particles in a concentrated mixture. The Fermi momentum

   $$p_F = (3\pi^2 N_3)^{\frac{1}{3}} \hbar \sim (\hbar/A_0) x^{\frac{1}{3}} \tag{1.2}$$

   According to (1.1), it is possible to study the mixture in the "classical" Boltzmann regime, $T >> T_F$, the degenerate case $T << T_F$, and in all the intermediate region. Therefore, ^{3}He- He II mixtures are the unique objects: I do not know of any other systems which exist in such a wide temperature range –from the ultra-quantum degenerate regime $T << T_F$ up to the extremely "classical" Boltzmann $T >> T_F$ case.

4. **"Zero" (quantum) energy**

$$E_0 = \frac{\hbar^2}{mA_0^2}$$

This is the energy of zero ($T = 0$) quantum oscillations of particles. Depending on what one substitutes for the mass (m_3 or m^*) and A_0, the value of E_0 for ^{3}He systems is somewhere between 1 K and 10 K.

5. **Debye temperature**

$$\theta \sim \frac{\hbar c}{A_0} \sim 20K$$

c is the sound velocity. This, and the

6. **Elastic energy**

$$mc^2 \sim \theta(\theta/E_0) \sim 150K$$

characterize bulk dense ^{4}He. The Debye temperature is a semi-quantum parameter, while the elastic energy is the purely classical one. The hierarchy of the above energies is:

$$mc^2 >> \hbar\theta >> E_0 \geq T, T_F$$

while the ratio T/T_F may be arbitrary depending on the temperature and ^{3}He concentration.

Single ^{3}He impurity in He II

Superfluid ^{4}He at $T = 0$ is an absolutely uniform and homogeneous system without any excitations. What is more, it can be characterized by a macroscopic (stationary and homogeneous) wave function. He II at $T = 0$ is an example of non- trivial physical vacuum.

Therefore, the stationary states for any single impurity particles should be characterized – like in standard vacuum, "empty space" – by plane waves, i.e. energy and momenta states:

$$\psi_{imp} = const. \exp(-i\frac{\epsilon}{\hbar}t + i\mathbf{pr}/\hbar) \tag{1.3}$$

where the energy, ϵ, is a function of momenta, $\mathbf{p}$. Unlike the standard vacuum, ϵ should not necessarily be $p^2/2m$.

The wave functions for ^{3}He impurities should also have the form (1.3). Of course, the real question is

What is the form of $\epsilon(\mathbf{p})$ for ^{3}He impurities?

Unfortunately, the microscopic theory, at least at present, is unable to provide us with an adequate – taking into account a rather high accuracy of experimental data – answer. The reason is a very complicated spectrum of vacuum (He II) excitations, and our rather primitive knowledge of ^{3}He-He II interaction. The parameters of the spectrum contain both the direct interaction terms and all sequences of indirect diagrams/processes – processes mediated by He II phonons and rotons.

Then, may be, we can determine the exact spectrum $\epsilon(\mathbf{p})$ from comparison with some experimental data – assuming that we can perform extremely accurate measurements?

Unfortunately, again, the answer is NO. Below I will try to explain why.

Since our physical vacuum (He II) is absolutely isotropic, uniform and homogeneous, the spectrum of a single ^{3}He impurity can be expanded in even orders of the momentum $\mathbf{p}$:

$$\epsilon(\mathrm{p}) = -\Delta + \frac{p^2}{2M}\{1 + \gamma(\frac{p}{p_0})^2 + \xi(\frac{p}{p_0})^4 + \cdots\} \tag{1.4}$$

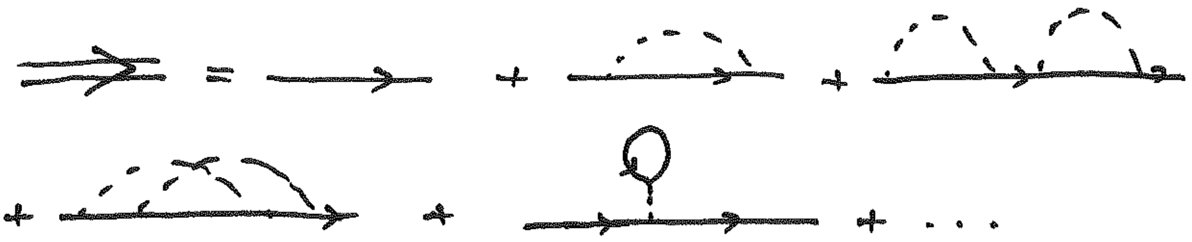
where I have chosen $p_0 = m_4 c \sim \frac{\hbar}{A_0}(m_4 c^2/\theta)$ as a scaling parameter, m_4 is the ^{4}He atomic mass, c is the sound velocity.

According to the current experimental data (at SVP):

1. $\Delta \sim 2.8$ K. The negative sign at Δ in eq.(1.4) means that the energy of the system lowers when a ^{3}He atom goes into ^{4}He thus ensuring the finite solubility of ^{3}He in ^{4}He at $T = 0$. At higher pressures Δ is somewhat larger.

2. The ^{3}He single-particle effective mass $M \approx 2.34 m_3$, m_3 being the mass of the bare ^{3}He atom. This enhancement in the ^{3}He effective mass means that the motion of a ^{3}He impurity effectively involves an accompanying motion of some "cloud" of ^{4}He atoms, or, in other words, excitation and absorption of ^{4}He phonon/rotons. The real propagator for a ^{3}He particle contains all sorts of

phonon diagrams like



Of course, the exact summation is impossible.

Curiously enough, at SVP

$$M \approx m_3 + m_4! \qquad (1.5)$$

meaning that, roughly speaking, a ^{3}He impurity quasi-particle carries in its motion a ^{3}He atom plus one ^{4}He atom. Of course, most probably, eq.(1.5) is a pure coincidence. At higher pressures M increases, and eq.(1.5) is not valid any more.

3. γ is unknown. It looks like γ is rather small (the "best" data – which are not accurate at all – claim that $\gamma \sim 0.15$ at SVP).
4. ξ and higher order coefficients are not known at all.

Ideal gas of ^{3}He quasi-particles

The basic idea how to determine the parameters of the single- particle energy spectrum (1.4) from experimental data is the following:

Suppose we perform very accurate experiments with extremely dilute mixtures. Then we might ignore all the effects caused by interaction of ^{3}He quasi-particles between themselves and which should be proportional to the (square of) ^{3}He concentration, and, therefore, are concentrationally small for dilute mixtures. We could measure several different properties of ^{3}He-^{4}He mixtures and compare the results with the calculations for an ideal gas of non-interacting ^{3}He quasi-particles with the spectrum (1.4) where the parameters Δ, M, γ, ξ, $\cdots$ should be considered as phenomenological parameters. This comparison (the best fit) would give us the values of the several parameters in the spectrum (1.4). Ideally, the more properties of ^{3}He-^{4}He we would measure (with more and more precision), the more of the parameters in (1.4) we would get.

Let us illustrate this idea on the example of, say, energy E of a mixture (the corresponding measurements are the measurements of heat capacity and of limiting ^{3}He concentration). If one may neglect the ^{3}He-^{3}He interaction at low ^{3}He concentrations, then the energy (per unit volume) of the mixture is phonon diagrams like

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$$E = E_0 + E_1 \tag{1.6}$$

where E_0 is the energy of pure ^{4}He, and E_1 is the energy of an *ideal* (non-interacting) *gas* of ^{3}He quasi- particles:

$$E_1 = \int \epsilon(\mathbf{p}) n(\mathbf{p}) d^3p/(2\pi\hbar)^3 \tag{1.7}$$

where $n(\mathbf{p})$ is the usual Fermi-Dirac distribution for ^{3}He fermions,

$$n(\mathbf{p}) = 1/[\exp(\epsilon(\mathbf{p}) - \mu)/T + 1], \tag{1.8}$$

$\epsilon(\mathbf{p})$ is the single-particle spectrum (1.4), μ is the chemical potential for ^{3}He particles.

The calculation of the integral (1.7) with (1.4) and (1.8) is a standard and trivial task. Before presenting the results, I want to point out that the proper parameterizations in degenerate, $T << T_F$, and Boltzmann, $T >> T_F$, cases, are different. In the degenerate cases the characteristic momenta of particles are the Fermi momenta $p \sim p_F$ (1.2), characteristic velocities are the Fermi velocities, $v \sim v_F \approx p_F/M$, and the energies – the Fermi energies, $\epsilon_F \approx p_F^2/2M$. Therefore, at very low temperatures, $T << T_F$, the energy (integral) (1.7) should look like

$$E_1 = -\Delta N_3 + \alpha_{11}\frac{p_F^2}{2M}N_3 + \alpha_{12}\gamma\frac{p_F^2}{2M}(\frac{p_F}{p_0})^2 N_3 \tag{1.9}$$
$$+\alpha_{13}\xi\frac{p_F^2}{2m}(\frac{p_F}{p_0})^4 N_3 + \cdots$$

where $\alpha_{11} = 3/5$, $\alpha_{12} = 3/7$, $\alpha_{13} = 1/3, \ldots$. Since p_F is proportional to the concentration $x^{\frac{1}{3}}$, we can rewrite (1.9) as

$$E_1 A_0^3 = -\Delta x + E_0(\beta_{11}x^{5/3} + \beta_{12}x^{7/3} + \beta_{13}x^3 + \cdots) \tag{1.10}$$

where the dimensionless coefficients β_{ik} are trivially related to M, γ, $\xi, \ldots$ and α_{ik} in (1.9).

As we see, the energy (1.10) at $T << T_F$ is the expansion in the concentration (to the power 1/3), $x^{1/3}$, with the coefficients being determined by the coefficients in the single- particle spectrum (1.4).

At higher temperatures, in the Boltzmann $T >> T_F$ domain, the characteristic momenta, velocities and energies are the thermal momenta, velocities and energies:

$$\epsilon \sim T, p \sim p_T = (Tm)^{\frac{1}{2}}, v \sim v_T = (T/m)^{\frac{1}{2}} \tag{1.11}$$

Therefore, at these temperatures the expression for the energy (1.7) is an expansion in temperature:

$$E_1 A_0^3 = E_0 x \left[-\frac{\Delta}{E_0} + \gamma_{11}\frac{T}{E_0} + \gamma_{12}(\frac{T}{E_0})^2 + \gamma_{13}(\frac{T}{E_0})^3 + \cdots\right] \tag{1.12}$$

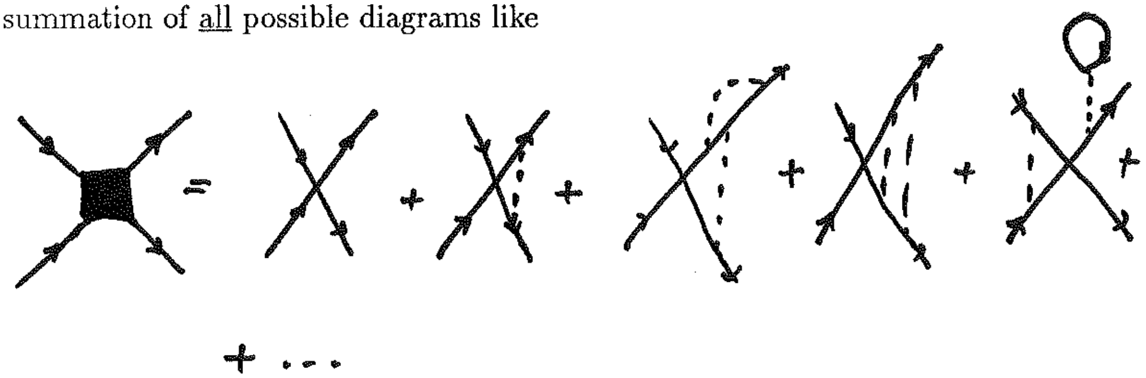
where the dimensionless coefficients γ_{ik}, like β_{ik} in (1.10), are determined by M, γ, ξ, ..., and are of the order of 1.

The idea is to measure the energy, $E = E_0 + E_1$, as accurately as possible, compare it with the expansions (1.10) or/and (1.12) [or the corresponding expressions in the intermediate temperature range], determine from the best fit the coefficients β_{ik} or/and γ_{ik}, and thus get the data on Δ, M, γ, ξ, ... It looks reasonable, and *one may hope to get very accurate data on the parameters of the single- particle spectrum (1.4) assuming that the experimental accuracy is desirably high. Unfortunately, this statement is absolutely wrong, and it is hopeless to try to go beyond a certain approximation (see below) despite a possible (even infinitely) high accuracy of experiments.* The reason is the

1.2 Interaction between ^{3}He quasi-particles

We should not take into account any terms in the expansions (1.10), (1.12) which are smaller than the neglected terms which contain the interaction.

Of course, there is always some interaction between ^{3}He quasiparticles. It is impossible to have exact microscopic description of the interaction just for the same reasons as for the exact single-particle spectrum. The interaction contains not only "direct" ^{3}He-^{3}He processes, but also the part mediated by phonons, rotons, etc. The summation of all possible diagrams like



is absolutely unfeasible without any unambiguous and real small parameters [all suggested parameters are very speculative and absolutely inapplicable for precise descriptions with a definite accuracy].

What makes the situation even worse, is a not very small ratio of particles velocities v to the sound velocity c (i.e. to the velocities of phonons), $v/c \leq 1$. As a result, the motion of ^{3}He particles between the moments of emission an absorption of intermediate (virtual) phonons is not negligible, and the phonon-mediated interaction of ^{3}He particles should exhibit strong retardation effects [similar to, but stronger than retardation effects in interaction of relativistic particles in electrodynamics]. This makes the possibilities of precise theoretic descriptions even much less probable.

Therefore, we have to deal with the interaction in the same way as with the single-particle spectrum: we have to introduce a reasonable/proper parameterization, and hope to find the corresponding parameters from some precise experiments.

Taking the interaction into account, the expression for the energy of the solution (1.6) should be rewritten as where the dimensionless coefficients γ_{ik}, like β_{ik} in (1.10), are determined by M, γ, ξ, ..., and are of the order of 1.

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Taking the interaction into account, the expression for the energy of the solution (1.6) should be rewritten as

$$E = E_0 + E_1 + E_2 + E_3 + \cdots \tag{1.13}$$

where E_0 is, as before, the energy of pure ^{4}He, E_1 is the single-particle (ideal gas) contribution (1.7), E_2, $E_3, \ldots$ are the contributions of the two-particle, three-particle, etc. interactions. Schematically, the terms E_2, E_3, ..., can be written as

$$E_2 = \int \frac{d^3p d^3p'}{(2\pi\hbar)^6} U_{12}(\mathbf{p}, \mathbf{p}') n(\mathbf{p}) n(\mathbf{p}') \tag{1.14}$$

$$E_3 = \int \frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi\hbar)^9} U_{123}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) n(\mathbf{p}_1) n(\mathbf{p}_2) n(\mathbf{p}_3) \tag{1.15}$$

and so on, where U_{12} is the potential of two-particle interaction, U_{123} – for three-particle processes, etc.

Note, that though we can have pretty reasonable (qualitative) ideas on how U_{12} might look like, the terms like U_{123} and higher are beyond any reach. This gives us a real limit on the accuracy for all, even future, theories.

What we can do, is to parameterize U_{12} in eq.(1.14), calculate the integral, and compare the results with eqs.(1.10), (1.12).

The easiest way of parameterization is a simple general Taylor expansion:

$$\begin{aligned} U_{12}(\mathbf{p}_1, \mathbf{p}_2) &\equiv U_{12}(p_1, p_2, \cos\chi) = \\ &= U_0(\chi) + U_1(\chi)\frac{p_1}{p_0} + U_2(\chi)\frac{p_2}{p_0} + U_3(\chi)\frac{p_1^2}{p_0^2} + \\ &+ U_4(\chi)\frac{p_2^2}{p_0^2} + U_5(\chi)\frac{p_1 p_2}{p_0^2} + \cdots \end{aligned} \tag{1.16}$$

Since at low temperature $T << T_F$, the characteristic values of both p_1 and p_2 are of the order of $p_F \sim (\hbar/A_0)x^{\frac{1}{3}}$, and the integral (1.14) with U_{12} (1.16) has the structure of the density expansion in $x^{\frac{1}{3}}$

$$E_2 A_0^3 = E_0 x^2 (\tilde{\beta}_{11} + \tilde{\beta}_{12} x^{\frac{1}{3}} + \tilde{\beta}_{13} x^{\frac{2}{3}} + \tilde{\beta}_{14} x + \cdots) \tag{1.17}$$

where the dimensionless coefficients $\tilde{\beta}_{ik}$ are given by simple combinations of angular integrals of the function U_i (1.16) scaled by E_0. By the order of magnitude, $\tilde{\beta}_{ik}$ in eq.(1.17), like β_{ik} in eq.(1.10), are of the order of 1.

It is obvious, that the concentration expansion for three- particle contributions, E_3, starts from the term x^3. *Therefore the terms with $\tilde{\beta}_{14}$ and higher in eq.(1.17), and β_{13} and higher in eq.(1.10) should be neglected when making comparison even the most precise experimental data:* these terms are of the same order (or even smaller) as some already neglected contributions. This means, that we can hope to learn from the experiment only the parameters Δ, M, γ in the single-particle spectrum (1.4), and only the terms up to the second order in the two-particle interaction (1.16).

At first glance, it seems that the situation in the non- degenerate (Boltzmann) case is much more encouraging: here the expansion in concentration (interaction) differs from the expansion in momenta which is an expansion in temperature.

In this case, E_1 has the from (1.12), E_2 (1.15) with U_{12} (1.16) gives us

$$E_2A^3 = E_0x^2\left(\tilde{\gamma}_{21} + \tilde{\gamma}_{22}(\frac{T}{E_0})^{\frac{1}{2}} + \gamma_{23}(\frac{T}{E_0}) + \tilde{\gamma}_{24}(\frac{T}{E_0})^{\frac{3}{2}} + \cdots\right) \tag{1.18}$$

while E_3 starts from the terms x^3. Therefore, one might hope to go to extremely low concentrations and try to recover, first, all terms in E_1 (which are proportional to x), then $-E_2$ (proportional to x^2), etc.

However, this is not realistic for both fundamental and practical reasons. The fundamental reason is related to the changes in phonon spectrum due to presence of ^{3}He impurities. The phonon contribution to the energy is also an expansion in ^{3}He concentrations and energies:

$$E_{ph} = \theta(\frac{T}{\theta})^4\left(1 + x[\delta_{11} + \delta_{12}\frac{T}{E_0} + \cdots] + x^2[\tilde{\delta}_{11} + \tilde{\delta}_{12}\frac{T}{E_0} + \cdots] + \right) \tag{1.19}$$

Since the ^{3}He/phonon renormalizations (δ_{ik} are unknown, the expansion for E_1 (1.12) is meaningful only up to the term γ_{13}, while (1.18), in case of vanishingly small concentrations becomes completely meaningless.

And the practical limitation is due to the fact that in order to be able to have the reliable data on impurity concentrations x up to the terms $(T/E_0)^n$, we have to perform precise measurements at concentrations

$$x < (T/E_0)^n$$

which is not easy, since $E_0 > 1$ K and T should be low enough in order not to involve thermal phonons.

Conclusion

It is possible to extract – even from the most precise experiments – only the first three terms in the single-particle spectrum and two-particle interaction.

We have an additional problem:

We know how to parametrize the spectrum: eq.(1.14) looks quite reasonable. But how to deal with an interaction in order

1. not to go beyond the accuracy?
2. not to use ugly expressions like (1.16)?

The answer is that we should not operate with the interaction potential at all.

According to statistical physics and kinetic theory, an intelligent way to introduce the interaction is to not do deal with an interaction potential, but to work with the exact scattering amplitude (or $T-$ and $S-$matrices, or vertex part/function, etc.). In our case it is not only proper, but also makes the things easier!

In our case of ^{3}He-^{4}He mixtures, a real small dimensionless parameter is

$$pA_0/\hbar \leq 1 \tag{1.20}$$

According to standard quantum mechanics, the scattering amplitude of *slow* particles with a relative angular moment ℓ of orbital motion is proportional to

$$f_\ell \propto p^{2\ell}$$

According to above arguments, we can reliably introduce only the first three terms in (1.20) in the interaction. Being translated into the language of scattering amplitudes, this means that we should consider only the s-wave scattering with the amplitude

$$f_s = -a\left[1 + c(\theta,\phi)p^2 + O(p^4)\right], \tag{1.21}$$

and the p-wave scattering with the amplitude

$$f_p = b_0 \cos\phi p^2[1 + O(p^2)] \tag{1.22}$$

Therefore, we have only constants a (the scattering length), b_0 and the function $c(\theta,\phi)$.

Unfortunately, because of retardation effects in the s-wave channel, the structure of $c(\theta,\phi)$ is not very clear, and this term – though it is a legitimate term! – should better be neglected. By the same reason, at least at present, one should not count on getting the data on b_0 from the experiments when the contribution of the s-wave scattering is not negligible. Therefore,

Conclusion

In most of the cases the only legitimate form of the spectrum is

$$\epsilon = -\Delta + \frac{p^2}{2M}[1 + \gamma(\frac{p}{p_0})^2] \tag{1.23}$$

and for the scattering length – a constant

$$f = -a \tag{1.24}$$

In cases when, like in high magnetic field (see below), the s-wave scattering becomes ineffective, the dominant role is played by the p-wave scattering with

$$f = b_0 p^2 \cos\phi \tag{1.25}$$

One can get from experiments only the values of Δ, M, γ, a, and, in high magnetic fields — b_0.

The accuracy is $p_F a/\hbar \sim x^{\frac{1}{3}}$ at low temperatures (unfortunately, $x^{\frac{1}{3}} >> x$; for example, when $x = 6.5\%$, $x^{\frac{1}{3}}$ is only ~ 0.4), or x and T/E_0 at much lower concentrations.

Note, that even if the scattering amplitude is a constant (1.24), this does not mean that the interaction potential is a constant or that the interaction function (see below) will not have any angular dependence.

2 Main properties of ^{3}He-^{4}He mixtures

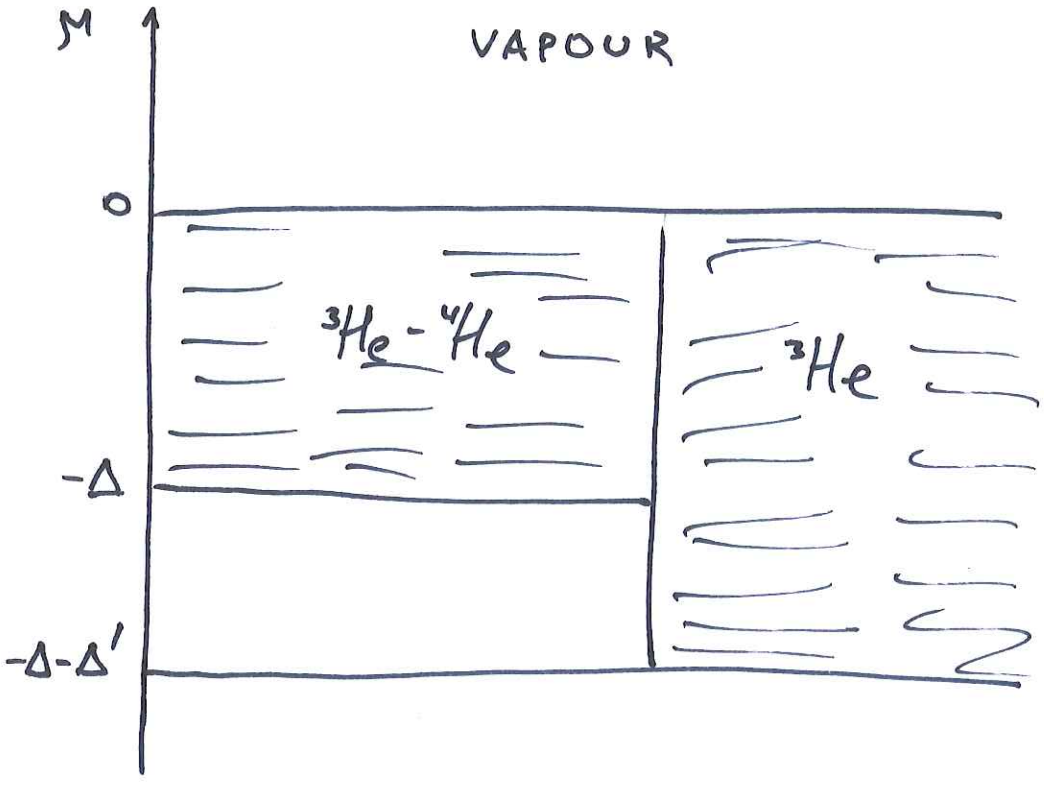
2.1 Thermo- and hydrodynamics

If one wants to have a simple (but, of course, not very accurate) estimate of any thermodynamic property, the easiest thing to do is to make an ideal gas estimate for a gas of ^{3}He quasi- particles. I will illustrate it on the example of a limiting concentration (demixing concentration) for the mixture at $T = 0$.

At zero (SVP) pressure one has at equilibrium ^{3}He-^{4}He mixture, ^{3}He liquid and ^{3}He vapour. Therefore the chemical potentials for ^{3}He particles in all three phases are equal

$$\mu_{3v} = \mu_{34} = \mu_3 \tag{2.1}$$

On the other hand, the chemical potential of a degenerate Fermi liquid/gas at $T = 0$ is equal to its Fermi energy. Schematically, the picture corresponding to (2.1) looks like this:



At $T = 0$ all the ^{3}He energy levels are filled up to $E_F = \mu$ ($= 0$ the chemical potential of vapour at $T \to 0$) for both, ^{3}He and ^{3}He-^{4}He. The difference between ^{3}He and ^{3}He-^{4}He is due to different densities of state (i.e., effective masses m^* for ^{3}He quasi-particles in a mixture and m_3^* in pure ^{3}He are different), and different positions of lowest levels (binding energies), which are $-\Delta$ and $-\Delta - \Delta'$. Now we can rewrite eq.(2.1) as

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At zero (SVP) pressure one has at equilibrium ^{3}He-^{4}He mixture, ^{3}He liquid and ^{3}He vapour. Therefore the chemical potentials for ^{3}He particles in all three phases are equal

$$\mu_{3v} = \mu_{34} = \mu_3 \tag{2.1}$$

On the other hand, the chemical potential of a degenerate Fermi liquid/gas at $T = 0$ is equal to its Fermi energy. Schematically, the picture corresponding to (2.1) looks like this:

At $T = 0$ all the ^{3}He energy levels are filled up to $E_F = \mu$ ($= 0$ the chemical potential of vapour at $T \rightarrow 0$) for both, ^{3}He and ^{3}He-^{4}He. The difference between ^{3}He and ^{3}He-^{4}He is due to different densities of state (i.e., effective masses m^* for ^{3}He quasi-particles in a mixture and m_3^* in pure ^{3}He are different), and different positions of lowest levels (binding energies), which are $-\Delta$ and $-\Delta - \Delta'$. Now we can rewrite eq.(2.1) as

$$\begin{aligned} 0 &= -\Delta + \frac{p_{F43}^2}{2m^*} \equiv \\ &\equiv -\Delta + \frac{\hbar^2}{2m^*}(3\pi^2 N_3)^{\frac{2}{3}} \end{aligned} \tag{2.2}$$

$$\Delta' + \frac{\hbar^2}{2m^*}(3\pi^2 N_3)^{\frac{2}{3}} = \frac{\hbar^2}{2m_3^*}(3\pi^2 N_{30})^{\frac{2}{3}} \tag{2.3}$$

where m^* and m_3^* are the ^{3}He effective masses in a mixture and in pure ^{3}He, N_3 and N_{30}- ^{3}He densities in dilute and dense phases. In principle, *eqs.(2.2), (2.3) may be considered as definitions* of Δ and Δ'.

In the ideal gas approximation for a mixture, one can use the spectrum (1.23) without γ, and assume that the effective mass of ^{3}He quasiparticles practically does not depend on ^{3}He concentration and, therefore, is equal to its limit M at $x \to 0$:

$$\Delta = \frac{\hbar^2}{2M}(3\pi^2 N_3)^{\frac{2}{3}}$$

i.e. the limiting concentration of the mixture is equal to

$$A_0^3 N_3 = x = \frac{1}{3\pi^2}\left(\frac{\Delta}{\hbar^2/2MA_0^2}\right)^{\frac{3}{2}} \tag{2.4}$$

or, starting from (2.3),

$$x = \left\{\frac{M}{m_3^*}(N_{30}A_0^3)^{\frac{2}{3}} - \frac{\Delta'}{\hbar^2/2MA_0^2}\frac{1}{(3\pi^2)^{\frac{2}{3}}}\right\}^{\frac{2}{3}} \tag{2.5}$$

where A_0^3 is the atomic volume in ^{3}He-^{4}He mixtures. Since x is relatively small while both terms in the curly brackets have "normal" orders of magnitude, these two terms are close to each other which gives a reasonable estimate for Δ'. The main inaccuracy in (2.4), (2.5) was related to an assumption $m^* \approx M$.

However, since our approximation for the interaction (1.24) – which is the only one adequate for the spectrum (1.23) – is not very complicated, it is not very difficult to go beyond the ideal gas approximation, and to include the interaction.

Here one encounters for the first time a very important difference between degenerate ($T << T_F$) and non-degenerate (Boltzmann) $T << T_F$ gases:

While for dilute Boltzmann gases the interaction function is proportional to the exact scattering amplitude (T-matrix), for degenerate dilute gases (and Fermi liquids) the interaction function is somewhat more complicated.

Let us illustrate it on the example of our (constant) scattering amplitude (1.24). In dilute Boltzmann gas the free energy F with the T-matrix (1.24) is equal to

$$\begin{aligned} F &= F_{40} + F_{id} + \frac{4\pi a\hbar^2}{M}\int n(\mathbf{p})n(\mathbf{p}')\frac{d^3p d^3p'}{(2\pi\hbar)^6} \\ &= F_{40} + F_{id} + F_{int}, \qquad F_{int} = \frac{\pi a\hbar^2}{M}N_3^2 \end{aligned} \tag{2.6}$$

where F_{40} is the free energy of pure ^{4}He, F_{id} is the free energy of the ideal gas of ^{3}He quasiparticles with the spectrum (1.23), and for the interaction correction F_{int} we were able to take the T-matrix out of the integral (cf. eq.(1.14)) benefiting from the independence of the amplitude (T-matrix) of momenta.

In a degenerate gas (or, what is the same, low-density Fermi liquid) the situation is more complicated because

- the Fermi-liquid interaction is not the T-matrix (or the vortex function Γ^K) but rather a somewhat different quantity, Γ^ω, which coincides with Γ^K (or $\hat{T}$) only in the linear approximation.
- the difference between the interaction potential and the T-matrix leads not to x^3 terms as for a Boltzmann gas (2.6), but to much smaller – and not negligible – terms $x^2 \cdot x^{\frac{1}{3}}$.

Therefore, the proper expression for the free energy (or energy, since $E = F$ at $T = 0$) of a nearly ideal Fermi gas with the single-particle spectrum (1.23) and two-particle interaction with the amplitude (1.24) has the form

$$\begin{aligned} E = E_0 + \int \left[-\Delta + \frac{p^2}{2M}\left(1 - \gamma \frac{p^2}{p_0^2}\right)\right] n(\mathbf{p}) \frac{2d^3p}{(2\pi\hbar)^3} + \\ + \frac{4\pi a \hbar^2}{M} \int n(\mathbf{p}) n(\mathbf{p}') \frac{d^3p d^3p'}{(2\pi\hbar)^6} - \\ - \frac{32\pi^2 a^2 \hbar^4}{M} \int \frac{n(\mathbf{p_1}) n(\mathbf{p_2}) n(\mathbf{p_3})}{p_1^2 + p_2^2 - p_3^2 - p_4^2} \frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi\hbar)^9} \end{aligned} \tag{2.7}$$

The main difference between (2.6) and (2.7) is the third term in (2.7) which would be beyond the accuracy for the Boltzmann gas.

The first two terms in (2.7) are exactly the same as in eq.(2.6). If one wants to have a uniform and general description for the whole temperature interval from ultralow to high temperatures, one should restrict oneself to these two terms. Then one would have a very general description, but the price would be a not very high accuracy everywhere. If one wants to have a more accurate description, one should always keep in mind that the

main corrections for different temperature intervals have very different origins. The main correction in one of the temperature regions looses its importance in another one, becomes illegitimate and beyond the accuracy, and should not be taken into account at all.

I deliberately spent so much time and put such an emphasis on the question of accuracy. The thermodynamic calculations themselves are pretty standard and technically not very sophisticated, while understanding of their accuracy is much less trivial.

With the expression (2.6), (2.7) for the thermodynamic potentials, we can calculate all the thermodynamic quantities directly. However, for the degenerate case (2.7) it might be more useful tutorially and sometimes more convenient, to calculate first the Fermi liquid function (the Landau function), and then calculate all the quantities using the standard expressions of the Fermi liquid theory.

The Fermi liquid function (the Landau function) is, by definition, the second (variational) derivative of the energy (in our case (2.7)) over the distributed function

$$F_{\sigma\sigma'}(\mathbf{p},\mathbf{p}') = \frac{\delta^2 E}{\delta n_\sigma(\mathbf{p})\delta n_\sigma(\mathbf{p}')} \tag{2.8}$$

where I – for the first time – have explicitly included the spin indices/variables $\vec{\sigma}$ into the distribution function $n_{\vec{\sigma}}(\mathbf{p})$ and into the Landau F- function.

The calculation of derivatives (2.8) for the energy (2.6) is trivial: of course, the second derivative of the terms E_0 and the first integral (linear in n) is zero, the second integral (quadratic in n) provides a constant, and the second derivative of the third integral leaves only one n in the integrand making the integration easier:

$$\begin{aligned} F_{\sigma\sigma'}(\mathbf{p},\mathbf{p}') &\equiv F_{\sigma\sigma'}(p_F,\theta) = \\ &= \frac{2\pi a\hbar^2}{M}(1-\vec{\sigma}\vec{\sigma}\,') + \\ &+ \frac{2\pi a\hbar^2}{M}2\lambda\left\{2+\frac{1-w^2}{w}\tanh^{-1}w-(1-w\tanh^{-1}w)\vec{\sigma}\vec{\sigma}\,'\right\}, \end{aligned} \tag{2.9}$$

where $\lambda \equiv p_F a/\pi\hbar \sim x^{\frac{1}{3}}$, $w \equiv \sin(\theta/2)$.

Several comments to eq.(2.9):

- Though the scattering T-matrix (amplitude (2.14) does not depend on momenta/scattering angles, the Landau F-function (the effective interaction function) strongly depends on angles θ between the momenta of interacting quasiparticles.

- Though the "bare" scattering amplitude (1.14) corresponds only to the s-wave scattering, the resulting effective interaction (2.9) contains all angular harmonics (including, e.g., the p-wave channel).

- Higher angular harmonics in the effective interaction (2.9) contain an additional small factor $\lambda = p_F a/\pi h$ in comparison with the main (constant) terms for the effective s-wave scattering.

And a comment of a different type:

- In principle, one can use a Fermi liquid like approach even at higher temperatures, in the domain of Boltzmann gases (the so- called concept of "statistical quasi-particles"). However, to do so for ^{3}He-^{4}He mixtures, one should restrict oneself only to the first (constant) term in eq.(2.9); the rest would be beyond the accuracy (in some sense, this is a re-formulation of the comment made after eq.(2.7)).

With the explicit expression for the Fermi liquid function (2.9), the calculation of all thermodynamic quantities becomes trivial, and should be performed according to the standard rules of the Landau theory of Fermi liquids. As a result, all the observables become concentration-dependent through the parameter $\lambda = p_F a/\pi\hbar \sim x^{\frac{1}{3}}$ in (2.9). I will not go here into details of the calculations, and will give below only few results (at $T \to 0$) mostly for the thermodynamic quantities which do not contain large masking contributions coming from pure ^{4}He.

1. Effective mass m^* of (dressed) Fermi liquid quasi- particles is related to the (bare) mass M of a single ^{3}He excitation as

$$m^* = M\left[1 + \frac{8}{15}\lambda^2(7\ln 2 - 1) + 2\gamma\frac{p_F^2}{p_0^2}\right] \tag{2.10}$$

2. The osmotic pressure in a system of two cells (one with pure ^{4}He, $x = 0$, and another with a mixture ^{3}He-^{4}He, $x \neq 0$) connected by a superleak permeable only for a superfluid component ^{4}He, is equal to

$$\Pi = \frac{2}{5}\frac{p_F^2}{2M}N_3\left[1 + \frac{5}{3}\lambda + \frac{8}{21}\lambda^2(11 - 2\ln 2) - \frac{10}{7}\gamma\frac{p_F^2}{p_0^2}\right] \tag{2.11}$$

3. The normal density is

$$\rho_n = MN_3\left[1 + 2\gamma(p_F/p_0)^2\right] \tag{2.12}$$

4. The second sound velocity (the velocity of hydrodynamic sound in the gas of ^{3}He impurity quasi-particles) is

$$s_2^2 = \frac{p_F^2}{3M^2}\left[1 + 2\lambda + \frac{8}{15}\lambda^2(11 - 2\ln 2) - 4\lambda\frac{p_F^2}{p_0^2}\right] \tag{2.13}$$

5. The magnetic susceptibility is

$$\frac{\chi_{id}}{\chi - \chi_4^{(0)}} = 1 - 2\lambda - \frac{16}{15}\lambda^2(2 + \ln 2), \tag{2.14}$$

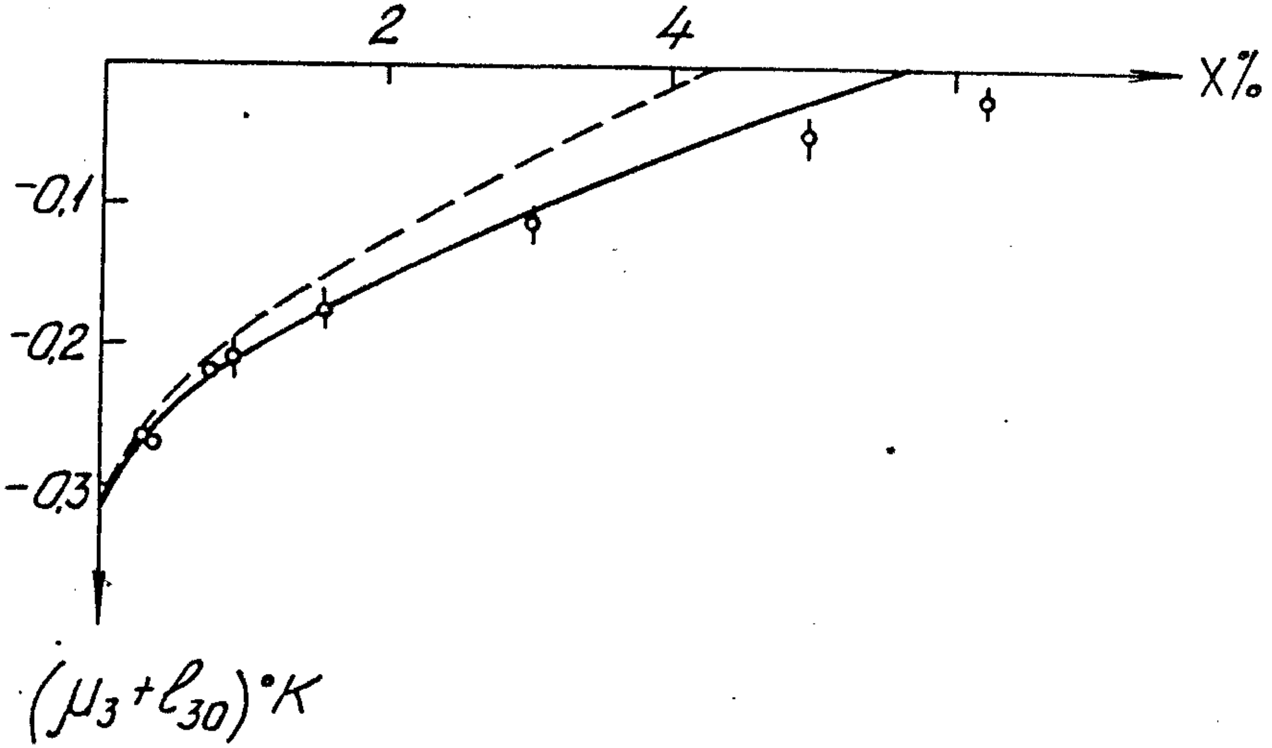
   where $\chi_4^{(0)}$ is the diamagnetic susceptibility of ^{4}He, χ_{id} is the susceptibility of an ideal gas.

Next 5 curves allow one to compare experimental data on chemical potential, effective mass, osmotic pressure, magnetic susceptibility, and spin diffusion with the above formulae. For the sake of comparison, the dashed curve for the chemical potential shows its value in the ideal gas approximation without any interaction corrections.

erate ideal Fermi gas. All the experimental [illegible]

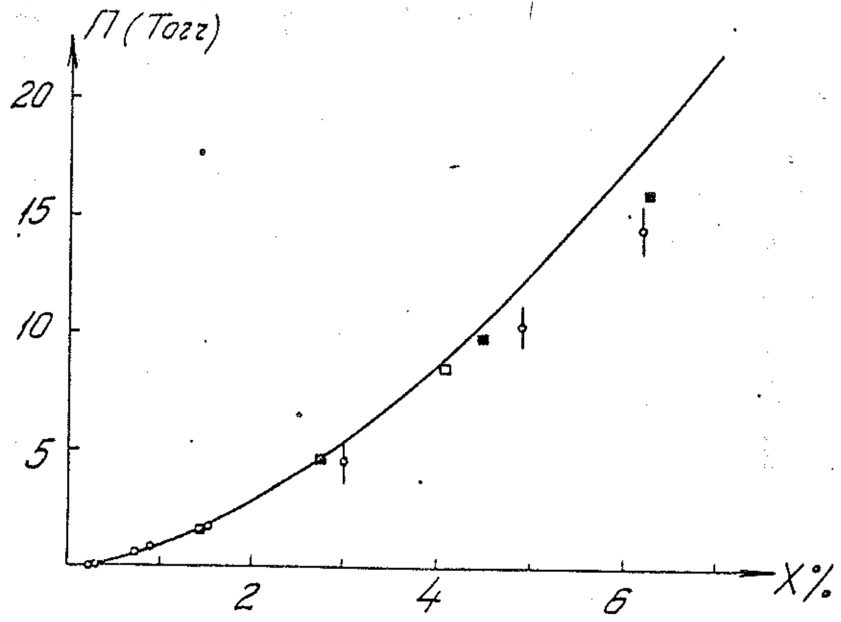
[illegible]

Fig. 1



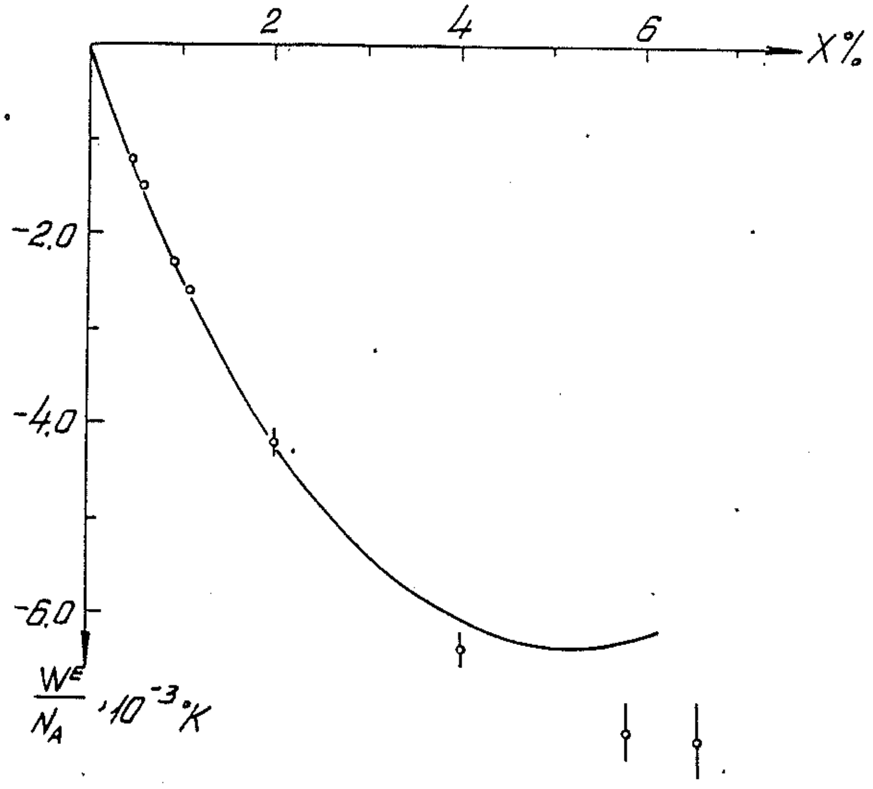
tration dependence of the chemical potential of ^{3}He. The dashed curve corresponds to n ideal gas. $x=N_3/(N_3+N_4)$. ○, the experimental results of Seligmann *et al.* (1969).

Fig. 2



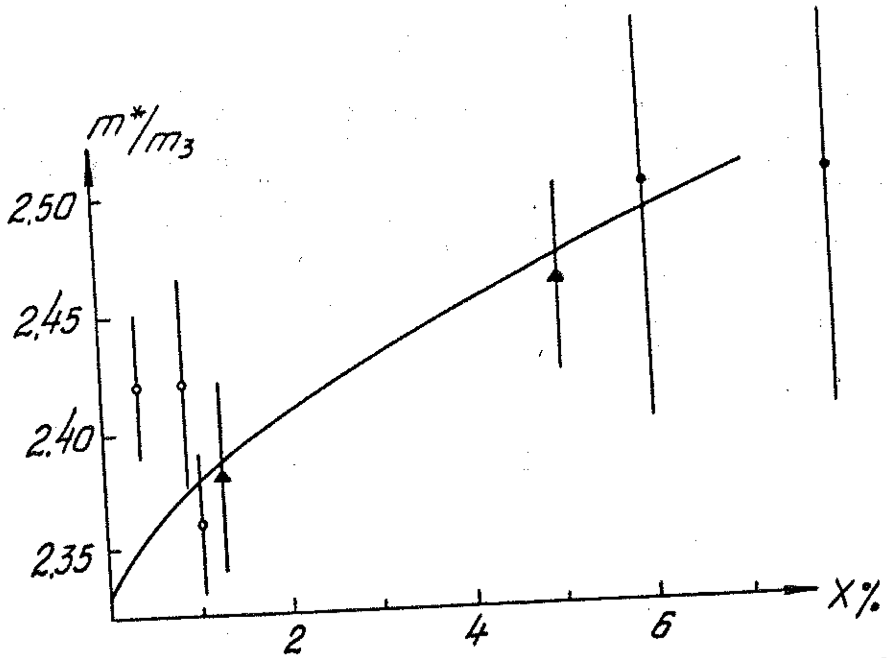
Osmotic pressure in a degenerate solution. □, ■, data of Landau *et al.* (1969, 1970); ○, data of Seligmann *et al.* (1969).

Fig. 3



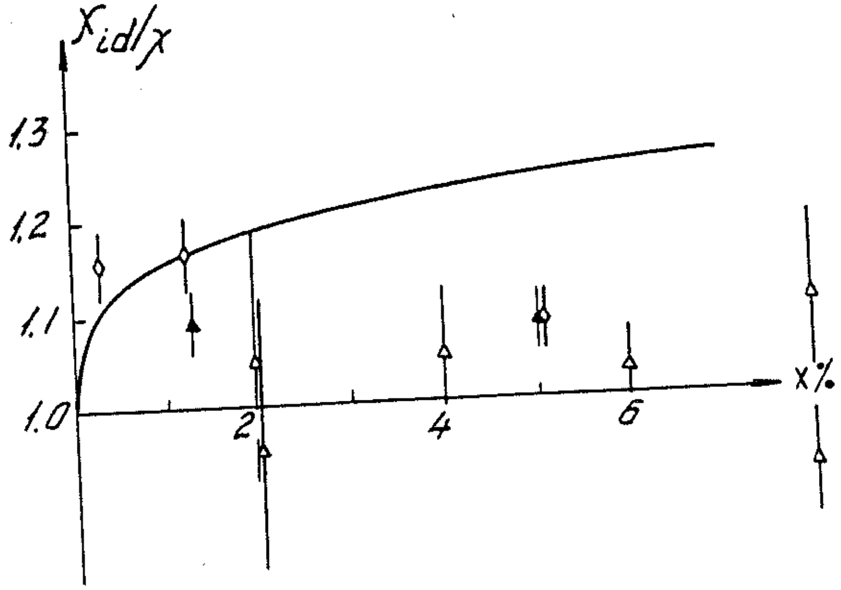
Excess enthalpy of a degenerate solution (N_A is Avogadro's number). ○, data of Seligmann *et al.* (1969).

Fig. 4



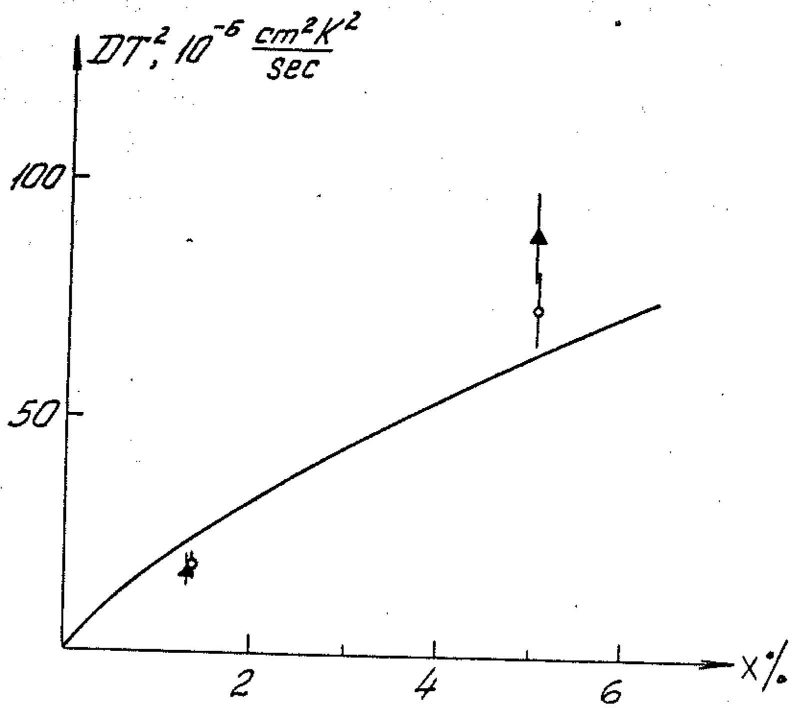
Total effective mass of impurity quasiparticles. ▲, data of Anderson *et al.* (1966 b); ○, ●, data of Seligmann *et al.* (1969) and Edwards *et al.* (1965).

Fig. 5



Magnetic susceptibility of a solution. △, data of Husa *et al.* (1966 a, b); ▲, data of Anderson *et al.* (1966 b); ◇, data of Ahonen *et al.* (1976) for the relative susceptibility normalized to the value for $x = 1{\cdot}3\%$ calculated using eqn. (2.4.15).

Fig. 6



Concentration dependence of spin diffusion coefficient. ▲, data of Anderson *et al.* (1966 b); ○, data of Murdock *et al.* (1978).

2.2 Kinetics: transport and HF phenomena

To a large extent, the theory of transport in ^{3}He-^{4}He mixtures is formally the same as for transport properties in a (dilute) gas of ^{3}He quasi-particles while the influence of superfluid ^{4}He is reduced to background (vacuum) renormalizations of parameters. Therefore all the transport coefficients (viscosity η, thermal conductivity κ, spin diffusion D, and sound attenuation α'') are given by standard gas expressions and are inversely proportional to a^2. For example,

$$\eta = \frac{1}{24\pi^3} \frac{p_F^5}{M^2 a^2 T^4} \cdot 0.81 \cdot (1 + 0.74\lambda) \tag{2.15}$$

Basically, the above theory is a straightforward generalization of standard approaches to slightly non-ideal degenerate or Boltzmann gases. The influence of the ^{4}He background was hidden in and reduced to vacuum renormalizations of the effective mass and scattering length. It looks like one can forget about ^{4}He the moment one learns the values of M and a.

The presence of ^{4}He becomes much more important when one tries to approach collective phenomena (oscillations). Here the coupling between ^{4}He and ^{3}He becomes much more complicated because all the ^{3}He effective parameters like Δ, M, a, γ, etc. depend explicitly on the density of the ^{4}He superfluid background, N_4. Therefore, any changes in ^{4}He density lead immediately to changes in state of ^{3}He component resulting in coupling of the equations, i.e. in mutual drag effects and coupling of oscillations.

This coupling is due to changes, e.g. in Δ or M, with changes δN_4 of N_4:

$$\begin{aligned} \delta\Delta &= (\partial\Delta/\partial N_4)\delta N_4, \\ \delta M &= (\partial M/\partial N_4)\delta N_4, \end{aligned} \tag{2.16}$$

and so on. Most of such derivatives are absolutely unknown except for the most important one

$$\frac{\partial\Delta}{\partial N_4} \sim 0.28 \frac{m_4 c^2}{N_4} \tag{2.17}$$

where c is the sound velocity in pure ^{4}He. Fortunately enough, the main (in ^{3}He concentration) terms for coupling of oscillations correspond to the mechanism (2.17) as a sole source of coupling. Unfortunately, this leads to some additional limitations for the accuracy of calculations.

Now I will briefly list all possible collective modes:

1. First sound: low-frequency acoustic (density) oscillations close to the usual first sound in pure ^{4}He. The ^{3}He component is responsible for small changes in the sound velocity, but is completely (at low temperatures) responsible for the sound attenuation (via viscosity and thermal conductivity).

2. Second sound. Roughly speaking, acoustic (density) oscillations in a gas of ^{3}He quasi-particles. At $T = 0$, $s_2 \approx v_F/\sqrt{3}$.

3. HF modes cannot propagate in the Boltzmann region because of strong collisionless (Landau, or Cherenkov) damping.

4. Zero sound modes cannot propagate even in degenerate ^{3}He subsystem because of

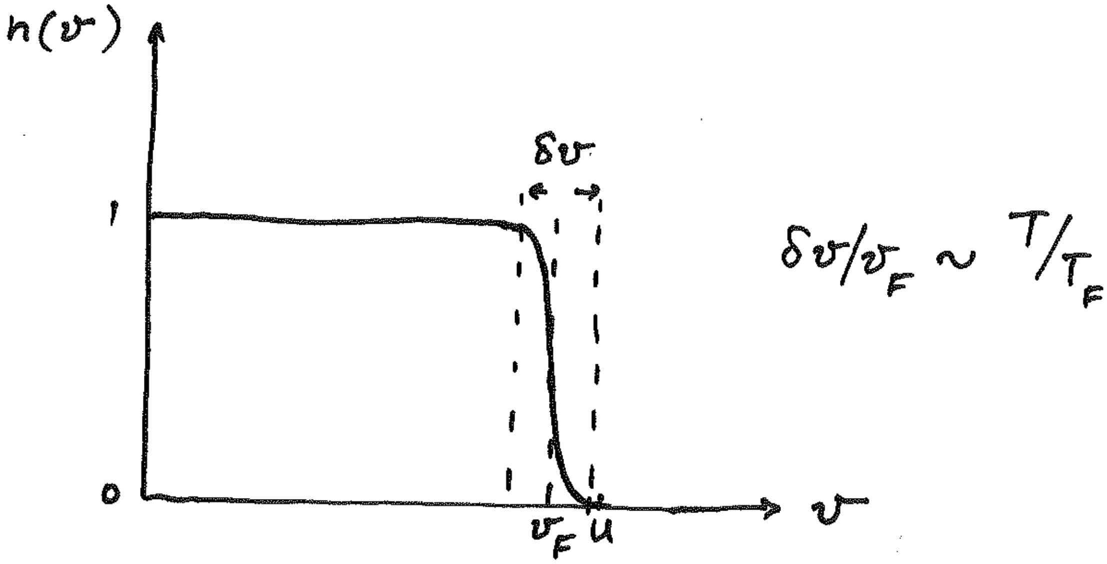
   (a) attraction (unlike pure ^{3}He!) in the s-wave channel.

   (b) coupling to ^{4}He, and small values of higher harmonics of the Landau F-function.

5. The only possibility is the symmetric $m = 0$ ("longitudinal") spin zero sound (= longitudinal spin wave). However, since the Fermi liquid interaction is concentrationally small, the velocity of the wave, u, is exponentially close to the Fermi velocity v_F,

$$u - v_F = v_F \exp(-1/\lambda) \tag{2.18}$$

Therefore, the real danger for an existence of such a mode comes from the collisionless (= Landau = Cherenkov) damping. Such an attenuation is proportional to the number of particles $n(u)$ moving in phase with the wave (thus being able to absorb a quantum of the wave). Therefore, in order not to experience a strong damping, the velocity of the wave, u, should be outside (see the figure)



of the region of the thermal smearing of the Fermi step. Mathematically this is equivalent to the following inequality for the temperature

$$T << T_F \exp(-1/|\lambda|), \tag{2.19}$$

or, numerically,

3. HF modes cannot propagate in the Boltzmann region because of strong collisionless (Landau, or Cherenkov) damping.

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   of the region of the thermal smearing of the Fermi step. Mathematically this is equivalent to the following inequality for the temperature

   $$T << T_F \exp(-1/|\lambda|), \tag{2.19}$$

   or, numerically,

$$T[mK] << 4.2 \cdot 10^3 x^{\frac{2}{3}} \exp(-\frac{3.6}{x^{\frac{1}{3}}|a|}) \tag{2.20}$$

with a in Å.

At first glance, eq.(2.20) looks like an ordinary restriction for the temperature. The situation is not that simple. In reality, the temperature (2.20) lies below the superfluid transition temperature for ^{3}He subsystem (see below). It is known that the longitudinal spin wave cannot propagate below such a transition: the Cooper pairing is accompanied by the formation of pairs of particles with opposite spins thus making impossible the formation of quanta of longitudinal spin waves which are the correlated states of pairs of particles with parallel spins.

However, as we will see in the next section, that the situation is not hopeless: one still can observe such a mode in very weak magnetic fields slightly above the superfluid transition in ^{3}He component.

2.3 Desirable future experiments

The only reliable information we have is the information at zero (SVP) pressure. According to experimental data

$$\begin{aligned} 2.3m_3 \leq M &\leq 2.6m_3 \\ 0.5\mathring{A} \leq -a &\leq 1.2\mathring{A} \\ \gamma &\lesssim 1 \end{aligned} \tag{2.21}$$

The accuracy is not high, especially for a. It looks like M and $(-a)$ increase with pressure. One needs more accurate experiments preferably at low ^{3}He concentrations ($x < 0.1\%$) and temperatures.

From this point view the most desirable experiments are

1. Any accurate experiments at higher pressures!
2. Osmotic pressure: a good experimental possibility especially for M, and less – for a and γ.
3. Sound attenuation, viscosity and
4. Spin diffusion can provide reliable and accurate information on a.
5. Second sound velocity: a good way to obtain M.
6. Magnetic susceptibility: most of old experiments are inaccurate and unreliable.
7. Bulk and wall values of T_1: the systematic study of T_1 is very important for any experimental program on spin- polarized mixtures.

All these experiments will not (most probably) lead to an observation of some new and unexpected effects, but should be a base for any consistent future program.

The key factor in obtaining the best values of parameters is the ^{3}He concentration. One must always keep a proper balance: the accuracy of experiments decreases with decreasing concentration while the theoretical description becomes more accurate.

3 Superfluidity of ^{3}He in ^{3}He-^{4}He mixtures

3.1 Transition temperature: is there any hope?

The honest answer is that nobody knows. One should not believe in any numbers (and in most of the cases – even in orders of magnitude) for T_c given by theorists. This does not mean that the underlying theories are dead wrong: even if some theory is basically correct and provides one with a physically adequate picture, the accuracy of numerical estimates for T_c is not even low – it is simply unknown.

The situation now is somewhat analogous to what was going on before the experimental discovery of the superfluid transition in pure ^{3}He in early seventies. Now, as then, different predictions for T_c differ by several orders of magnitude, and fill (probably, with normal distribution) the interval from 10^{-8} K up to the temperatures just slightly below the current record (whatever it is) in ultra-low temperatures. What is more, like in pure ^{3}He, even after the ^{3}He superfluidity in mixtures will be observed, we will eventually have a perfect description of the corresponding superfluid phases, but the calculations of T_c will, most probably, remain unreliable.

As of now, we have a wide range of approaches starting from exact statements (but with narrow range of applicability) to some very wild speculations. Below I will try to present a somewhat balanced picture.

At first, I will make several statements, and then will comment on them, and discuss some consequences.

First, what is so special about the ^{3}He superfluidity in ^{3}He-^{4}He mixtures? Why should we be so excited about it except for a somewhat selfish reason that it is one of the very few unclear phenomena at ultra-low temperatures? To answer this question, we should understand why the corresponding superfluid phases will be very different from pure ^{3}He or ^{4}He.

The superfluid phases of ^{3}He in ^{3}He-^{4}He mixtures will be the phases:

- With two different condensates (^{3}He and ^{4}He). The only other system with two condensates is a neutron star where one has neutron and some proton condensates. The astrophysical implications are obvious. Other consequences of the presence of two condensates are
  - mutual drag effects
  - complicated quantization rules for vortices
  - new collective modes
- With superfluidity of non-charged fermions resulting in

  - the absence of the Meissner effect
  - unusual (paramagnetic) response to a magnetic field
  - lack of cyclotron motion, gauge invariance, etc.

- Without impurities, which opens the possibilities for different types of
  - unconventional pairing.

- But with the attraction in the s-wave channel which makes the system very different from pure ^{3}He, and closer to unconventional BCS superconductors.

Now I will explain why it is impossible to give a reliable estimate for T_c even assuming that one has a perfect understanding of the coupling processes.

^{3}He component of a mixture is a low density system even at highest concentrations/pressures. Therefore any reasonable theory should be, at least partially, a weak coupling theory. The equation for T_c in all weak coupling models/theories/approaches has the form

$$T_c \sim E \exp(-1/g) \tag{3.1}$$

where the cut-off (scaling) energy $E \lesssim T_F$, and the coupling constant g is given by some combination – depending on a theory – of interaction parameters and the density of states at Fermi energy.

Suppose, we have some uncertainty δg in the value of g. How will it be reflected in our prediction for T_c?

$$T_c(g-\delta g) = E \exp(-\frac{1}{g-\delta g}) \simeq T_c(g)(\frac{T_c(g)}{T_F})^{\delta g/g} \tag{3.2}$$

Therefore, the small correction δg will not change the <u>order of magnitude</u> of $T_c(g)$ only if

$$|\delta g/g| << T_c(g)/T_F \tag{3.3}$$

From all theoretical estimates (and our experimental experience as well) we know that

$$T_c/T_F \lesssim 10^{-3} \tag{3.4}$$

Therefore, *in order to predict even the correct order of magnitude of T_c, one has to understand all the details of the interaction (and know the density of states) with the accuracy higher than 0.1%!* Such a claim would be, mildly speaking, too optimistic.

Nevertheless, it is possible to make several definite statements.

1. The s-wave scattering amplitude of bare ^{3}He quasiparticles is negative, meaning that there is an attraction in the s-wave channel. Therefore, at low enough concentrations and magnetic field, the superfluid transition will be caused by the usual s-wave Cooper pairing and *will be described by the standard BCS theory with g in (3.1)*

   $$g \sim |a|p/\pi\hbar \sim x^{\frac{1}{3}} \tag{3.5}$$

2. The polarization of the spin system will suppress the s- wave pairing even at relatively low polarization. Then the next candidate for the transition will be the p-wave pairing, and phenomenologically the superfluid phase will resemble the A_1 phase of pure ^{3}He.

3. At low concentrations and polarizations the p-wave pairing will originate not from the p-wave contribution $b_0 p^2 \cos\phi$,

   $$g \sim (ap_F/\hbar)^3 \sim x, \tag{3.6}$$

   in the scattering amplitude of bare particles (1.22), but from p-wave-like contributions coming from the full interaction function originating from the s-wave channel with the interaction constant,

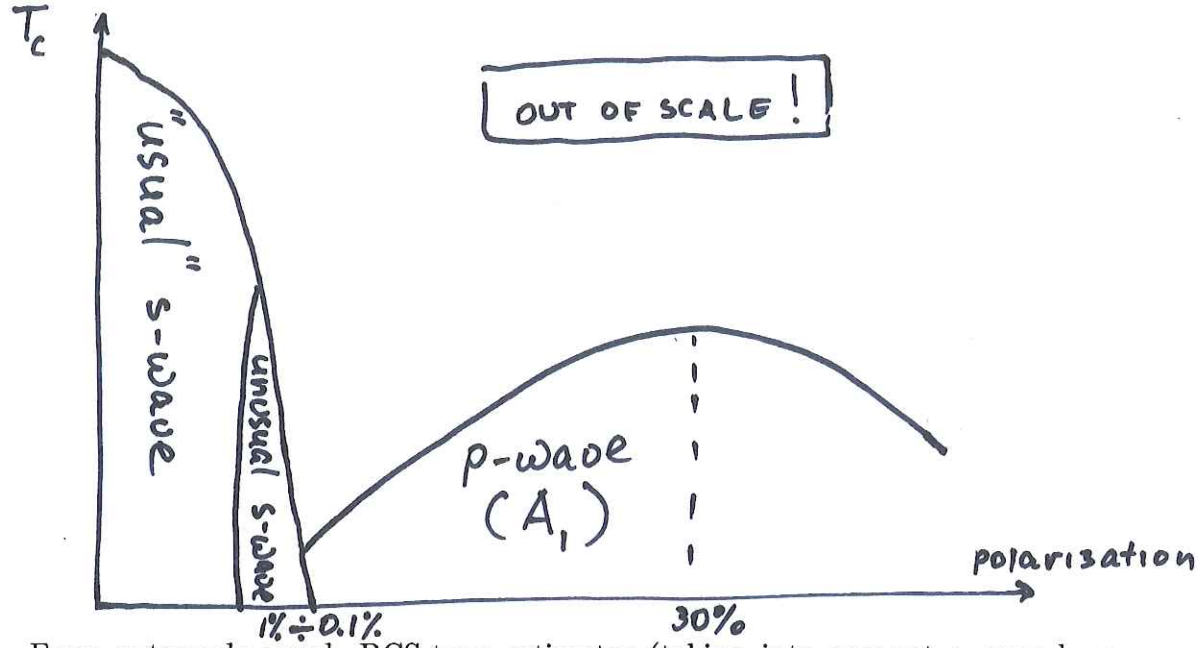
   $$g \sim (ap_F/\pi\hbar)^2 \sim x^{\frac{2}{3}}, \tag{3.7}$$

   larger than for the bare p-wave interaction (3.6). The appearance of the p-wave-like contributions from the s-wave interaction is somewhat similar to an appearance of all angular harmonics in the effective interaction in the second order s- wave interaction (2.9). (See also the comment after (2.9)).

4. At higher concentrations/pressures (but zero polarizations), I still believe in the dominant role of s-wave pairing (g (3.5) is still considerably larger than (3.6) or (3.7)), but, of course, this statement is an extrapolation and cannot be proved. The opposite statements are not (and cannot be) proved either, and are based on less reliable arguments, especially taking into account the above (Section 1 and 2) discussion on possible parameterizations of the interaction.

5. The p-wave transition temperature will have a non- monotonic dependence on polarization with a maximum somewhere around 30% polarization.

Qualitatively, the phase diagram, as I see it, should look like



From extremely rough BCS-type estimates (taking into account a very large change in limiting solubility of ^{3}He with polarization; see below) I cannot expect the p-wave transition temperature to be above 50 μK; most probably, it is much lower. However, we have 3 variables: concentration, pressure, and polarization, and nobody knows where exactly to look for the absolute maximum in T_c. I expect it to be at zero polarizations and at highest concentrations. Another (local) maximum is at polarizations about 30% and high concentrations.

Below I will mostly dwell in the left hand side (s-wave BCS side) of the above phase diagram.

The BCS prediction is

$$T_c = \frac{\gamma}{\pi}(\frac{2}{e})^{\frac{7}{3}} T_F \exp\{-\frac{\pi\hbar}{2p_F|a|}\} \tag{3.8}$$

or, numerically (at SVP)

$$T_c[K] = 0.71 x^{\frac{2}{3}} \exp(-\frac{1.8}{|a| x^{\frac{1}{3}}}) \tag{3.9}$$

with a in Å. The exact value of effective mass (i.e. T_F) is not that important (the uncertainty of several percent in the coefficient 0.71 is not very crucial), but, since the exponent $1.8/x^{\frac{1}{3}}|a|$ is rather larger, the whole equation (3.9) becomes very sensitive to even tiniest changes in a. Here it is even important what is x: N_3/N_4 or $N_3/(N_3+N_4)$!

Let us illustrate it on the basis of available information on a (see the table). Of course, one should expect that the values of a obtained from different types of experiments in different conditions by different groups should slightly differ from each

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Let us illustrate it on the basis of available information on a (see the table). Of course, one should expect that the values of a obtained from different types of experiments in different conditions by different groups should slightly differ from each other. But, as one can see, a reasonable discrepancy in the values of a, taken for their face values, leads to an enormous spread in T_c.

Source of information		$-a$, Å	T_c, mK $x = 0.03$)
Thermal conductivity	1967	0.83	0.06
Spin diffusion	1966	0.54	0.0015
	1981	0.52	0.001
Viscosity	1972	0.75	0.03
Spin echo	1984	0.5-0.7	0.0006-0.02
	1990	1	0.2

Is it hopeless to try to observe the transition? I cannot answer this question without more reliable and precise experimental data on a. But I want to remind that

the data obtained at very low concentrations are not very accurate (experiments are too difficult), and the data obtained at higher concentrations are not very reliable (the accuracy of interpretation is about $x^{\frac{1}{3}}$*).*

In general, the thermodynamics of the BCS-type s-wave transition for ^{3}He-^{4}He mixtures is the same as in the text- book BCS theory: the standard jump in heat capacity, decrease in magnetic susceptibility, etc. Actually, the ^{3}He subsystem in ^{3}He-^{4}He mixtures should be closer to the BCS model than any metal. Note, that one can observe the transition only when/if $T_c > \hbar/\tau^{mag)}$, where τ^{mag} is essentially T_1.

3.2 Three-velocity hydrodynamics

Below the transition the system has two different condensates, ^{3}He and ^{4}He. Therefore one should introduce 2 superfluid velocities (proportional to the gradients of the condensate wave functions) $\mathbf{v}_3$, $\mathbf{v}_4$, and a normal velocity $\mathbf{v}_n$:

$$\mathrm{v}_3 = \frac{1}{2m_3}\vec{\nabla}\phi_3 \quad , \quad \mathrm{v}_4 = \frac{1}{m_4}\vec{\nabla}\phi_4 \tag{3.10}$$

The critical velocity for ^{3}He

$$v_{3c} \sim T_c/p_F << v_F, v_{4c}$$

Since the change $\delta\epsilon(\mathbf{p})$ in ^{3}He energy in moving ^{4}He is (Galileo principle)

$$\delta\epsilon(\mathrm{p}) = (1 - \frac{m_3}{M})\mathrm{p}\mathrm{v}_4, \tag{3.11}$$

the superfluid motion of ^{4}He plays the same role for ^{3}He as a vector potential of magnetic field $\mathbf{A}$ for electrons in metal,

$$(1 - \frac{m_3}{M})\mathrm{v}_4 \Leftrightarrow \frac{e}{c}A \tag{3.12}$$

The mass conservation laws (continuity equations) for ^{3}He and ^{4}He have the standard form analogous to pure ^{4}He:

$$\begin{aligned} \dot\rho_3 &+ \mathrm{div}(\rho_3 \mathbf{v}_n + \mathbf{p}_3) = 0 \\ \dot\rho_4 &+ \mathrm{div}(\rho_4 \mathbf{v}_n + p_4) = 0 \end{aligned} \tag{3.13}$$

where, of course, the momenta of components are proportional to velocities, $\mathbf{v}_\mathrm{i} - \mathbf{v}_n$:

$$\begin{aligned} \mathbf{p}_3 &= \rho_{33}^{(s)}(\mathbf{v}_3 - \mathbf{v}_n) + \rho_{34}^{(s)}(\mathbf{v}_4 - \mathbf{v}_n) \\ \mathbf{p}_4 &= \rho_{43}^{(s)}(\mathbf{v}_3 - \mathbf{v}_n) + \rho_{44}^{(s)}(\mathbf{v}_4 - \mathbf{v}_n), \end{aligned} \tag{3.14}$$

with proportionality coefficients forming a 4-component matrix of superfluid densities. The elements of this matrix are

$$\begin{aligned} \rho_{33}^{(s)} &= (m_3^2/M)N_s, \\ \rho_{34}^{(s)} &= \rho_{43}^{(s)} = (M/m_3 - 1)\rho_{33}^{(s)} \\ \rho_{44}^{(s)} &= \rho_4 - (M - m_3)N_3 + (1 - m_3/M)^2 M N_s, \\ \rho_3^{(n)} &= m_3(N_3 - N_s), \\ \rho_4^{(n)} &= (M - m_3)(N_3 - N_s), \\ \rho^{(n)} &= M(N_3 - N_s), \end{aligned} \tag{3.15}$$

where N_s is the number of "superfluid" ^{3}He particles; N_s is determined by BCS formulae

In principle, the energy of ^{3}He quasi-particles depends on the superfluid velocities:

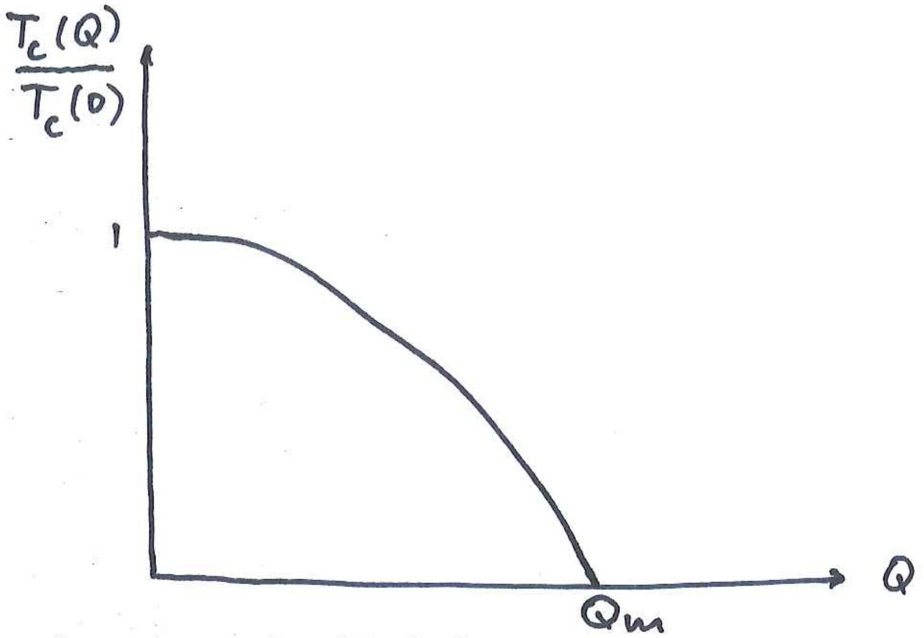
$$\delta\epsilon = \frac{m_3}{M}\mathbf{p}\mathbf{v}_3 + (1 - \frac{m_3}{M})\mathbf{p}\mathbf{v}_4 \tag{3.16}$$

Therefore, the quantity

$$\mathbf{Q} = 2m_3\mathbf{v}_3 + 2(M - m_3)\mathbf{v}_4 - 2M\mathbf{v}_n \tag{3.17}$$

plays the role of the *momentum of ^{3}He Cooper pairs.*

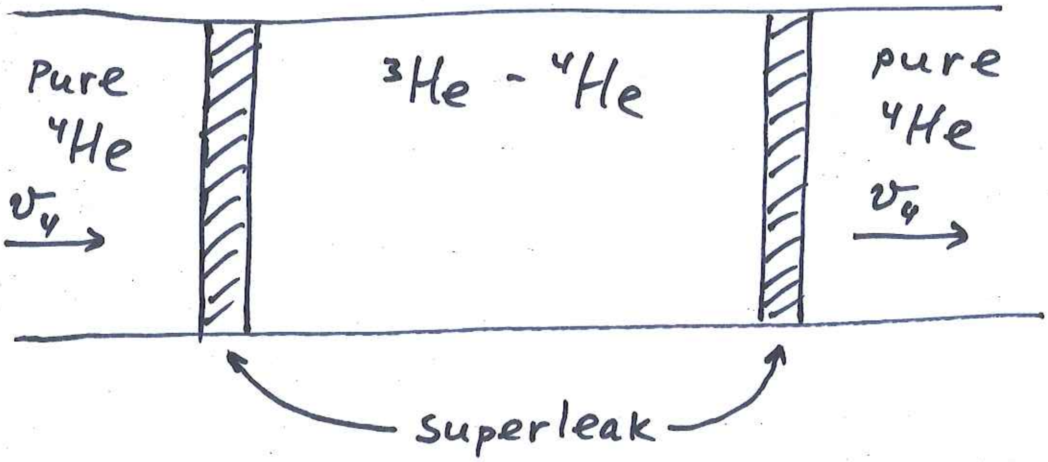
The transition temperature T_c strongly depends on Q:



where the maximum value of Q, Q_m is

$$Q_m = \frac{\pi T_c(Q=0)}{v_F}\exp(1-C), \tag{3.18}$$

C being the Euler constant. This effect might be observed in the experimental set-up like



3.3 Collective modes

In a system with two condensates the number of collective (Goldstone) modes is quite large. The temperature dependence of different sound velocities is presented in the figure.

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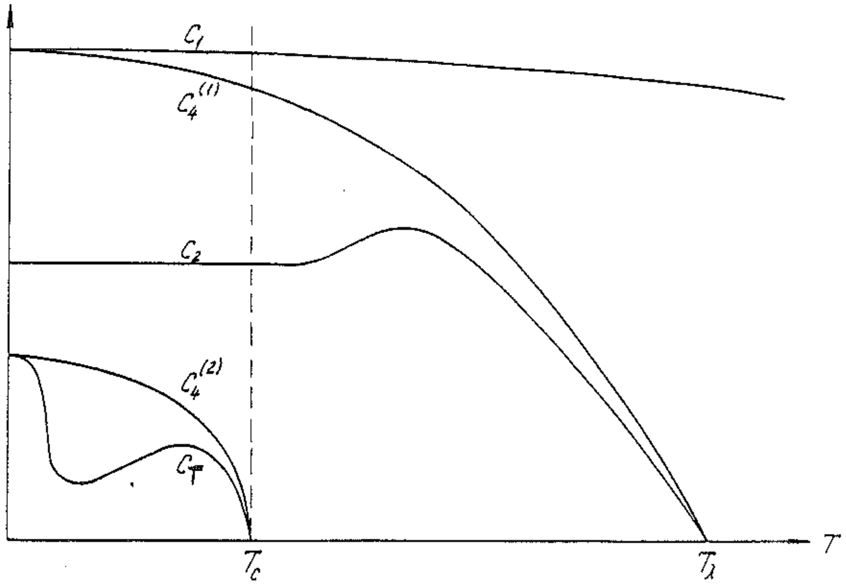
The standard classification of the modes is the following:

- c_1. First sound. Oscillations close to usual acoustic waves in pure He II. An interesting detail: at very low temperatures, $\omega\tau >> 1$, this mode is actually the so- called HF first sound, and not a hydrodynamic one. This mode is not very sensitive neither to the presence of ^{3}He, nor to the superfluid transition.
- c_2. Second sound. Oscillations similar to usual density (acoustic) oscillations in a gas of ^{3}He quasi-particles (or, at higher temperatures, in a gas of phonons). The mode disappears at T_λ. The mode is not very sensitive to T_c.
- c_T. Temperature waves in superfluid ^{3}He. The velocity is exponentially small in ^{3}He concentration,

$$c_T^2 = \frac{TS^2}{C}\frac{N_s}{MN_3N_n} \tag{3.19}$$

where S is the entropy, C - heat capacity.

- c_{24}. The fourth sound in superfluid ^{3}He.



Temperature dependence of propagation velocities for all hydrodynamic modes ($\omega\tau \ll 1$) in solutions (schematic).

3.4 Vortices

In systems with two superfluid velocities, the quantization rules on each linear vortex should be fulfilled separately for both $\mathbf{v}_3$ and $\mathbf{v}_4$:

$$v_3 = \frac{n_3}{2m_3}\frac{\hbar}{r}$$

$$v_4 = \frac{n_4}{m_4}\frac{\hbar}{r}$$

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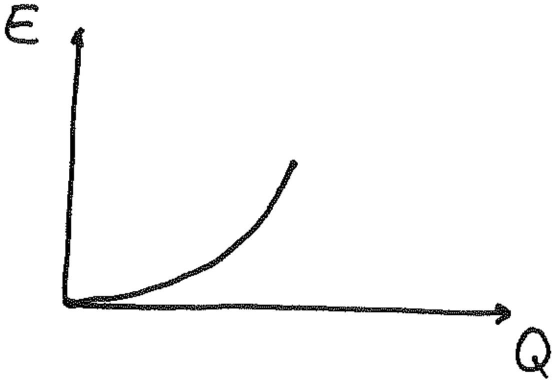
where n_3 and n_4 ("charges") should be integer. This means, that near the vortex the momentum of the Cooper pairs is non-zero

$$Q = \frac{\hbar}{r}\left|n_3 + \frac{2m_3}{m_4}\left(\frac{M}{m_3} - 1\right)n_4\right| \qquad (3.20)$$

On the other hand, the energy of the system is the monotonic function of Q:

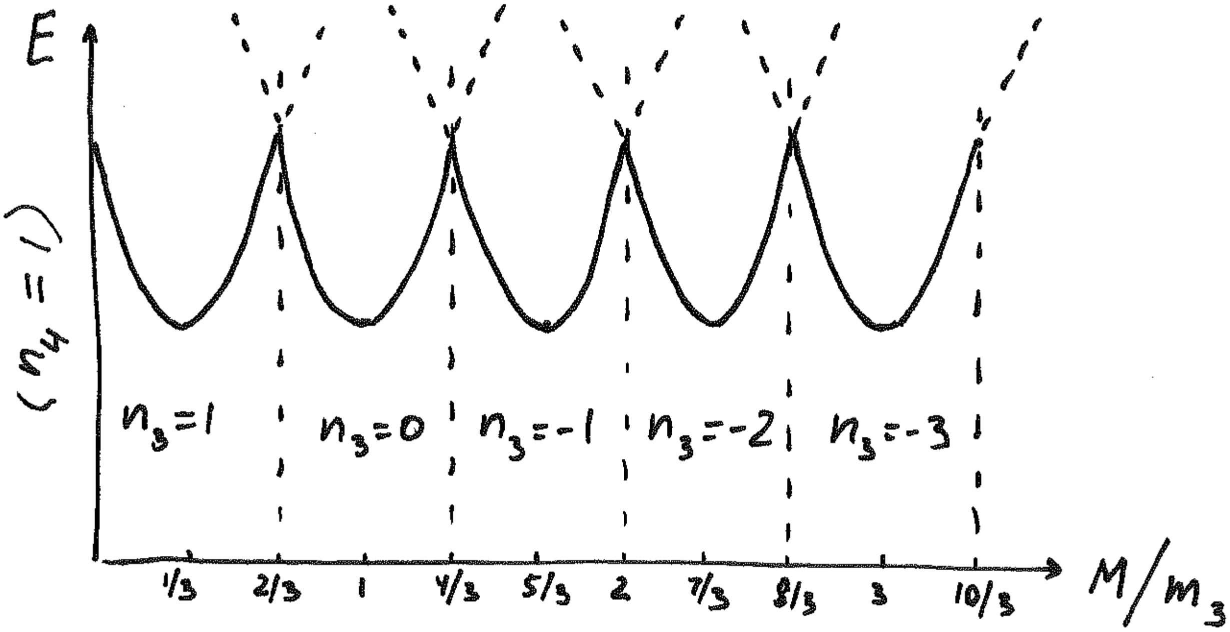


Therefore, the energy becomes the function of M, n_3, n_4:

$$E = E(M, n_3, n_4) \qquad (3.21)$$

The equilibrium vortices should have the charges n_3 corresponding to the minimum of the energy (3.21). As a result, the charges n_3 become *the functions of n_4 and ^3He effective mass, M.*

Let us consider the most common case $n_4 = 1$. Then the energies $E(Q)$ with different values of n_3 will have the following dependences $E_{n_3}(M)$:



Of course, only solid parts of the curves correspond to stable states. To observe transition from one curve to another, one should either measure the dependence of where n_3 and n_4 ("charges") should be integer. This means, that near the vortex the momentum of the Cooper pairs is non-zero

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Of course, only solid parts of the curves correspond to stable states. To observe transition from one curve to another, one should either measure the dependence of charges n_3 of stable vortices (vortices with the lowest energies) on M, or to follow the changes in the vortex lattice.

At SVP the value of M/m_3 is very close to 7/3 (1.5) meaning that the most stable vortices are the ones with $n_4 = 1$, $n_3 = -2$, and that we are extremely close to the minimum on the above curve $E(M)$. Strangely enough, the (very uncertain) experimental data show that at $P = 10$ atm. we are very close to another special point, $M/m_3 \approx 8/3$, i.e. close to the maximum of the curve. At even higher pressure, $p = 20$ atm, we are very close to the next minimum $M/m_3 \sim 2.9$, meaning that between 0 and 20 atm one should observe at least one phase transition in the vortex system, from $n_3 = -2$ to $n_3 = -3$.

This transition can be observed either as a phase transition for a single vortex line, or as a phase transition (restructuring) for the vortex lattice.

3.5 Spin polarization vs. superfluidity

The usual Cooper pairs are formed from the particles from the Fermi surface with opposite spins and opposite but equal momenta.

If we start to polarize the spin system making $N_\uparrow$ larger than $N_\downarrow$, the Fermi momenta for up and down spins, $p_\uparrow = \hbar(6\pi^2 N_\uparrow)^{\frac{1}{3}}$ and $p_\downarrow = \hbar(6\pi^2 N_\downarrow)^{\frac{1}{3}}$, become different. Therefore, for the particles it becomes more and more difficult with growing polarization to find a proper counterpart with an opposite spin and equal momentum to couple into the Cooper pair. However, finite temperatures provide sufficient uncertainty near the Fermi spheres to ensure the proper pairing if the difference in Fermi momenta for up and down spins is still small

$$\frac{p_\uparrow - p_\downarrow}{p_F} \leq \frac{T}{T_F} \tag{3.22}$$

If, however, the polarization is high enough to violate the inequality (3.20), the formation of Cooper pairs with zero momenta becomes absolutely impossible, and the only possible outcome is the formation of Cooper pairs with non-zero momentum Q.

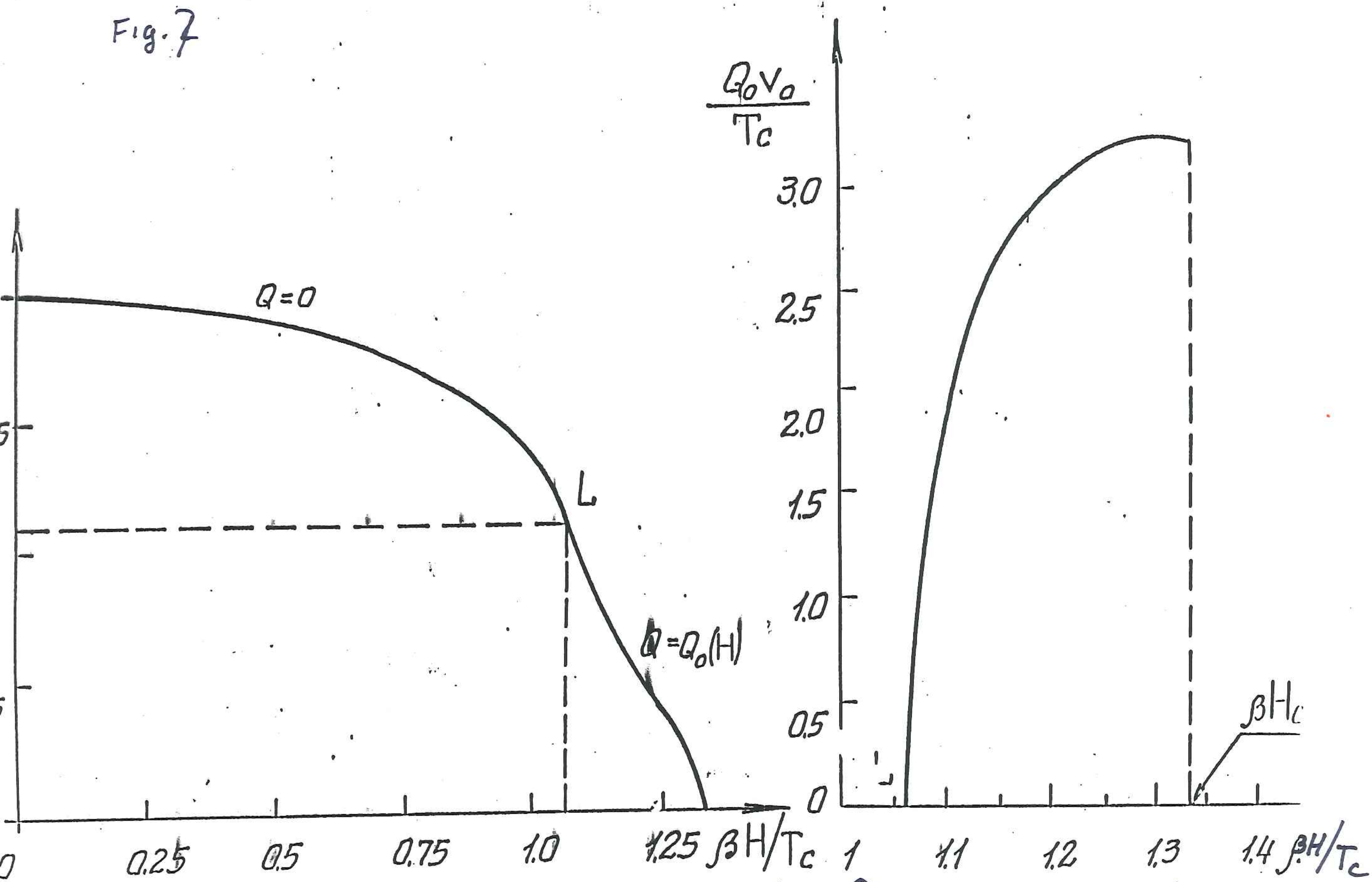
This all means that the transition temperature becomes smaller with increasing polarization (see the figure). Eventually, when the kinetic energy of the condensate associated with pair momenta Q will become comparable with the energy gap, Δ, the s-wave pairing will become completely impossible.

At very small polarizations, when Q is still zero, the major difference with a normal BCS phase will be some coupling of temperature waves with longitudinal spin oscillations since the equations in entropy and the magnetic moment M_2 become coupled via $\mathbf{v}_n$:

$$\begin{aligned} \dot{S} &+ S \operatorname{div} \mathbf{v}_n = 0 \\ \dot{M}_2 &+ M_2 \operatorname{div} \mathbf{v}_n = 0 \end{aligned} \tag{3.23}$$

The change in the transition temperature with polarization/magnetic field is given in the figure. The figure also shows the part of phase diagram where $Q \neq 0$.

Fig. 7



Dependence of the TRANSITION TEMPERATURE (AND Q) on magnetic field/POLARIZATION

3.6 Superfluid liquid crystals

When $Q \neq 0$, the superfluid phase becomes spatially inhomogeneous meaning that the order parameter (energy gap) Δ obtains a spatial structure

$$\Delta(\mathbf{r}) = \sum_m \Delta_m e^{i\mathbf{Q}_m \mathbf{r}/\hbar} \tag{3.24}$$

As a result, a superfluid mixture obtains some pseudo-crystalline structure. Near the transition point all the vectors $\mathbf{Q}_m$ in eq.(3.22) have the same value

$$|Q_m| = Q(T) \tag{3.25}$$

where $Q(T)$ is given in a figure. The expansion (3.24) involves higher and higher harmonics of Q (3.27) at lower temperatures.

The exact structure of the equilibrium phase (3.24) is still unknown. The most widely studied phases are

- The so-called Fulde-Ferrell phase

  $$\Delta = \Delta_0 e^{i\mathbf{Q}\mathbf{r}/\hbar} \tag{3.26}$$

  The transition into this phase is of the 2-d order. One of the major peculiarities of this phase is the equivalence of phase transformations

  $$\Delta \rightarrow \Delta e^{i\phi}$$

  to translations

  $$\mathbf{r} \rightarrow \mathbf{r} + \mathbf{u}$$

  which results in dynamical properties very different from other phases.

- The layered (2D) phase

  $$\Delta = \Delta_0 \cos(\mathbf{Q}\mathbf{r}/\hbar) \tag{3.27}$$

  The phase transitions may be of the second and of the first order depending on polarization. The phase shifts are not equivalent to translations.

- The cubic phase

  $$\Delta = \Delta_0 \left\{\cos Q_x/\hbar + \cos Qy/\hbar + \cos Q_z/\hbar\right\} \tag{3.28}$$

  The phase transition is of the first order.

In principle, such inhomogeneous superfluid phases might also be observed in case of superconductors. But in superconductors the magnetic field does not penetrate into the bulk (making it difficult to polarize the spin system), and unconventional pairs do not survive collisions with any impurities (making it impossible to polarize spins by introducing paramagnetic impurities into superconductor).

Above I have mentioned the order of the phase transition in the mean field approximation. One can also make an exact statement that

Due to fluctuations, the transition into the inhomogeneous phase is always of the first order.

The reason is quite simple. Since the transition occurs at finite momentum, **Q**, the phase space for fluctuations becomes quite large,

$$d^3\kappa \sim Q^2 dk$$

making the system look effectively like a one-dimensional one.

I will not go into details though some of them are very non- trivial and rather fascinating. I would only like to mention that in equilibrium

- the density of the system is constant and does not follow variations in $\Delta(\mathbf{r})$ (3.24).
- there is no current in equilibrium despite the inhomogeneities
- the density of magnetic moment is inhomogeneous and has the form of a spin density wave:

$$M_z = M_{z_0}\left\{1 + \sum_m \frac{|\Delta_m|^2}{T_c^2}\cos(2\mathbf{Q}_m\mathbf{r}/\hbar)\right\} \tag{3.29}$$

- the orientation of the system (of vectors $\mathbf{Q}_m$) is determined mainly by the walls, or, in very large systems, by dipole-dipole interaction and superfluid current.

The collective modes in phases (3.22) are also very different from those in homogeneous systems. Now the superfluid transition for ^{3}He gives rise not only for one additional hydrodynamic variable

$$\mathbf{v}_3 = \frac{\hbar}{2m_3}\vec{\nabla}Q_3 \tag{3.30}$$

but also for another one, namely, to displacements

$$\mathbf{r} \to \mathbf{r} + \mathbf{u}, \qquad \Delta(\mathbf{r}+\mathbf{u}) \neq \Delta(\mathbf{r}) \tag{3.31}$$

Therefore, in such a system one can observe in addition to spin- temperature modes (3.19), (3.23) (which, in turn, become highly anisotropic),

a new Goldstone mode – shear elastic waves, which are very unusual for liquids.

4 Spin-polarized ^{3}He↑-^{4}He liquid and gaseous mixtures; ^{3}He↑ gas

4.1 Thermo- and hydrodynamics

Now I would like to discuss the influence of spin polarization on the properties of ^{3}He-^{4}He mixtures (and ^{3}He gas). The choice of words "effects of spin polarization on ..." is deliberate, and is made not only because it sounds better than simple "magnetic properties". The underlying reason is that the spin polarization (spin alignment) is not necessarily related directly to an external (or internal) magnetic field:

The depolarization times (the time of longitudinal spin relaxation T_1) for helium systems may be very long providing the possibility of study of practically stationary non- equilibrium spin-polarized states with a degree of spin- polarization which is determined by a prehistory of a system rather than by a current intensity of an effective magnetic field.

Therefore, in many cases, the proper thermodynamic variable is a magnetic moment (or a degree of spin polarization), and not a magnetic field.

This also means that one has to be able to estimate characteristic times for different depolarization times, and to come up with reasonable suggestions for achieving high degrees of spin polarization.

The major part of ^{3}He magnetic interactions is of exchange origin and conserves the total magnetization. Two spin non- conserving processes are the very weak magnetic dipole-dipole interaction and magnetic relaxation on the walls. The wall processes are limited by the diffusion time L^2/D (L is the size of a sample, D is the spin diffusion coefficient), and the direct magnetic interaction with the wall. While the spin diffusion will be discussed later in detail, any general discussion of wall processes seems to be impossible: such processes depend on the material and structure of the wall, and on the possible presence of localized ^{3}He states near the wall. However, it is known that in many cases it is possible to suppress considerably the wall depolarization by coating the wall by some non-magnetic substance – most often, by molecular hydrogen and ^{4}He.

On the other hand, the dipole relaxation is a bulk property, and can be understood both experimentally and theoretically. In dilute systems like ^{3}He gas and ^{3}He-^{4}He mixture, a good estimate of the corresponding relaxation time T_1 is given by the expression

$$T_1 \sim \frac{\hbar^2 \Lambda^6}{\beta^4} (\frac{v}{\Lambda})^2 \frac{1}{N_3 v \Lambda^2} \tag{4.1}$$

which describes the probability of a spin-flip process under the action of magnetic dipole forces during the collision of two ^{3}He particles. Here the role of the size of a particle is played by its de Broglie wavelength, Λ.

At low temperatures,

$$T, T_F \leq E_0 \sim \hbar/ma^2$$

one should substitute $\Lambda \sim \hbar/mv$ in eq. (4.1):

$$T_1 \sim \frac{\hbar^4}{\beta^4 N_3 v m^2} \sim \frac{10^3}{x} \frac{1}{(T[K])^{\frac{1}{2}}}[s] \tag{4.2}$$

where in the Boltzmann region $T >> T_F$ the velocity $v \sim (T/m)^{\frac{1}{2}}$.

In the degenerate region, $T << T_F$, one should not only substitute the Fermi velocity,

$$v_F = \frac{\hbar}{M}(3\pi^2 N_3)^{\frac{1}{3}}, \tag{4.3}$$

into (4.2) instead of v, but also add an additional factor $(T_F/T)^2$ reflecting usual non-effectiveness of collisions in the degenerate regime

$$T_1 \sim \frac{\hbar^4}{\beta^4 N_3 v_F M^2}(\frac{T_F}{T})^2 \sim \frac{10^3}{x^{\frac{4}{3}}} \frac{10 x^{\frac{4}{3}}}{(T[K]])^2}, \qquad T[K] << x^{\frac{2}{3}} \tag{4.4}$$

Note, that $v_F \propto N_3^{\frac{1}{3}}$, $T_F \propto N_3^{\frac{2}{3}}$, and, as a result, T_1 does not depend on ^{3}He density N_3.

However, in case of strongly non-equilibrium polarization, when the degree of polarization is high and is much larger than its thermodynamic value determined by an external magnetic field, the phase-space arguments leading to a restricting factor $(T_F/T)^2$ are not valid any more. In this case the energy/momentum state of a flipped spin is practically not restricted, the factor $(T_F/T)^2$ disappears, and the relaxation time,

$$T_1 \sim \frac{\hbar^4}{\beta^4 N_3 v_F M^2} \sim \frac{10^3}{x^{\frac{4}{3}}}[s] \tag{4.5}$$

contains again an explicit density dependence.

At relatively high temperatures, $T >> E_0$, one should substitute into eq.(4.1) an atomic size a instead of Λ:

$$T_1 \sim \frac{\hbar^2 a^2}{\beta^4 N_3}(T/M)^{\frac{1}{2}} \tag{4.6}$$

The numerical estimates easily show that in many important cases the dipole time (4.1)-(4.5) is very long indeed, and that with a proper coating of walls one may expect to observe and study different overpolarized non-equilibrium states.

There are several ways how to polarize helium systems. The most straightforward method of those listed below is

- The so-called brute force technique: polarization by an external field. The degree of polarization $\alpha \sim 2\beta H/E$ may be noticeable in realistic fields $H \leq$ 100 Oe only if a characteristic magnetic energy E is of the order of or less than 10 mK (the magnetic moment of a ^{3}He atom ($\beta \sim 0.08$ mK/kOe). Therefore, this method is good only for solid ^{3}He (the Néel temperature $E_n \sim 1$ mK) and dilute ^{3}He-^{4}He mixtures with $T_F \leq 10$ mK at low temperatures $T \leq 10$ mK.

- The polarization of liquid ^{3}He and ^{3}He-^{4}He mixtures by the rapid melting of solid ^{3}He$\uparrow$ polarized by the brute force at low temperatures. Here the problems are associated with large heating during melting and with not very long relaxation time T_1 for liquid ^{3}He ($T_1 \leq 10$ min).
- The optical pumping for ^{3}He gas at room temperatures with consequent cooling. $N_3 \sim 10^{18}$ cm^{-3}, $\alpha \geq 80\%$, $T_1 > 3$ days, $T \leq 1$ K. The same group has managed to condensate this polarized ^{3}He$\uparrow$ gas into liquid.
- HF methods based on magnetic coupling of ^{3}He particles to the walls. This methods provide some enhancement in ^{3}He polarization over its equilibrium value, but are not very effective yet.
- The enhancement of ^{3}He concentration/polarization by taking away ^{4}He through a superleak (or by crystallization) out of ^{3}He$\uparrow$-^{4}He liquid mixture. These methods have not been tried yet.

Basically, these are two sources of polarization dependencies for physical characteristics of $^3He\uparrow$ *-* ^4He *mixtures:*

- Changes in characteristic kinetic energies/momenta/velocities of particles. This effect is much more noticeable for degenerate mixtures when the Fermi momenta follow the changes in (relative) densities of up and down spins, $N_{\uparrow(\downarrow)}$

  $$p_{\uparrow(\downarrow)} = \hbar(6\pi^2 N_{\uparrow(\downarrow)})^{\frac{1}{3}} \tag{4.7}$$

  In Boltzmann $T >> T_{\uparrow(\downarrow)} = p^2_{\uparrow(\downarrow)}/2m^*_{\uparrow(\downarrow)}$ case, this effect is not very noticeable since the characteristic kinetic energies for up and down spins are mostly determined by the temperature and are not very sensitive to polarization.

- Changes in effective interaction. Formally, neither the single particle spectrum (1.23),

  $$\epsilon_\sigma(\mathbf{p}) = -\Delta + \frac{p^2}{2m}[1 + \gamma(\frac{p}{p_0}^2] - \beta\vec{\sigma}\mathbf{H}, \tag{4.8}$$

  nor the two-particle interaction forces and scattering amplitude (1.24), (1.25)

  $$f_s = -a \tag{4.9}$$

  $$f_p = b_0 \cos\phi \tag{4.10}$$

  do not depend on polarization. But the effective interaction is sensitive not only to interaction forces, but also to a distribution of particles between spin-up and spin-down components:

The s-wave scattering of spin-$\frac{1}{2}$ fermions is effective (because of the Pauli principle) only for scattering of particles with different spins. Therefore, the polarization of the spin system (spin alignment) leads to a decrease in effective contribution of the s-wave scattering, and consequently to an increase of the role of the p-wave channel. Overall, the effective interaction falls drastically with polarization due to a suppression of the main (s-wave) interaction channel.

Since the s-wave scattering amplitude (4.8) does not depend on momenta, the first (main) term in the Fermi liquid function (2.9)

$$F_{\sigma\sigma'}(\mathbf{p},\mathbf{p}') = \frac{2\pi a\hbar^2}{M}(1 - \vec{\sigma}\vec{\sigma}\,') \tag{4.11}$$

remains the same, while the second one changes considerably (it contains certain integrals over distribution function). What is more, the spin structure of the F-function $F_{\sigma\sigma'}$ changes with polarization. It contains now not only terms with products $\vec{\sigma}\vec{\sigma}\,'$ as eq.(2.9) but the terms of the type

$$F_{\sigma\sigma'} = F_1 + F_2\vec{\sigma}\vec{\sigma}\,' + F_3(\mathbf{m}(\vec{\sigma}+\vec{\sigma}\,')) + F_4(\mathbf{m}\vec{\sigma})(\mathbf{m}\vec{\sigma}\,') \tag{4.12}$$

where $\mathbf{m}$ is the unit vector in the direction of magnetization. Of course, the terms F_3 and F_4 are small (in $p_F/\hbar \sim x^{\frac{1}{3}}$) in comparison with (4.10).

As I mentioned above, when the polarization is very high, the s-wave interaction is effectively suppressed, and the interaction originates mainly from the p-wave processes with the amplitude (4.9), and the corresponding Landau F-function is

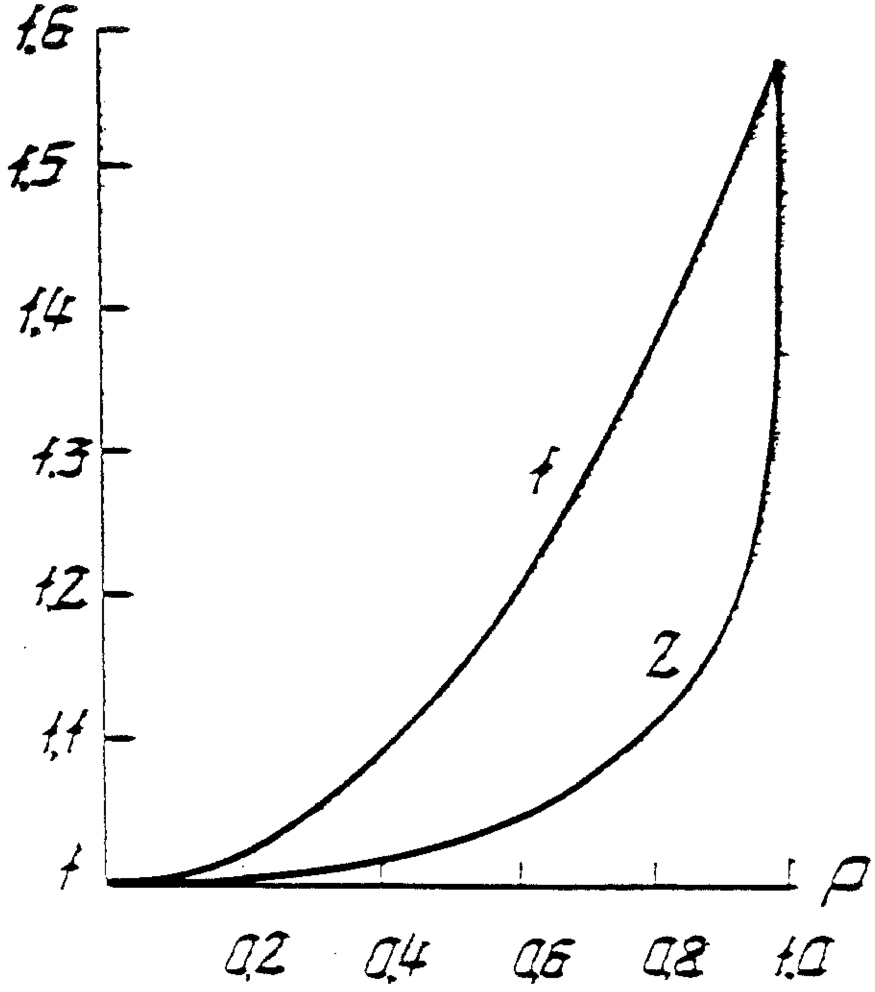
$$F = \frac{24\pi b}{M} p_F^2 \sin^2\frac{\theta}{2} \propto x^{\frac{2}{3}} \tag{4.13}$$

Therefore, the experiments at very high polarizations open the way to experimental determination of the constant b in the p-wave scattering amplitude (1.25), (4.9).

Most of the thermodynamic characteristic of ^{3}He↑- ^{4}He mixtures are not very sensitive to ^{3}He polarization because of the dominant role of background contribution of ^{4}He. There are few exceptions for which this background term does not mask the effects of polarization. I will discuss four of such effects.

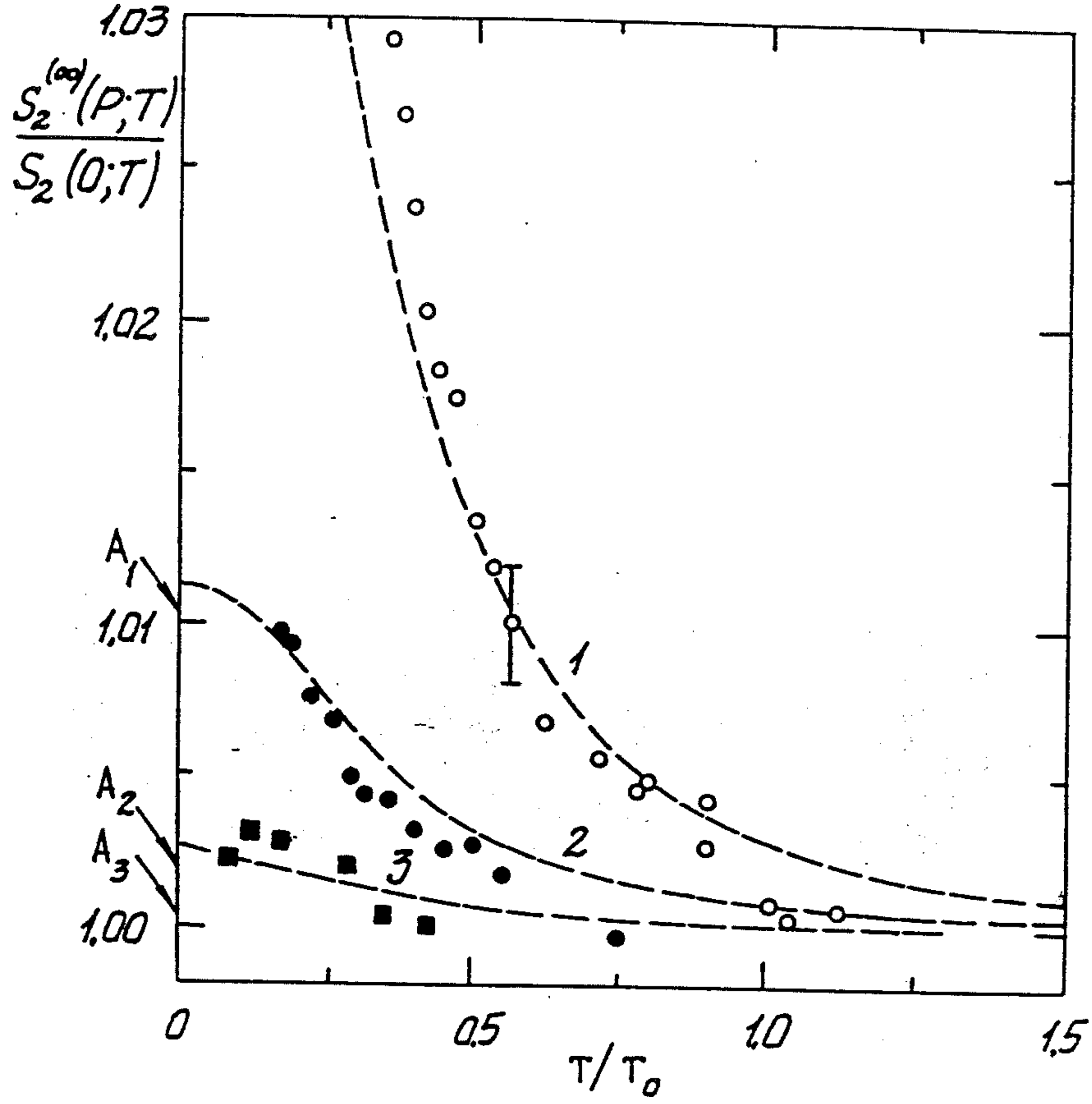
1. **Osmotic pressure.** The osmotic pressure – the pressure in the gas of ^{3}He quasi-particles – is proportional to the kinetic energy of the particles. Since in degenerate gas the kinetic energies strongly depend on polarization (4.6), the osmotic pressure changes considerably with polarization (see the figure). On the other hand, in the Boltzmann region the kinetic energies are affected by polarization only through small interaction correction, and polarization changes in osmotic pressure are very insignificant in comparison with a degenerate case.

   For exactly the same reason, the polarization effects are much more pronounced in degenerate case for



Polarization dependence of the relative change of the osmotic pressure $\Pi(P)$ curve 1), and of the second-sound velocities $s_2^{(\infty)^2}(P)/s_2^2(0)$, (CURVE 1) (cu $s_2^{(0)^2}(P)/s_2^2(0)$, (curve 2) in dilute degenerate solutions.

Fig. 8



perature dependence of the relative change of the second sound v
$_2(0; T)$. Experimental data and the calculations (in the ideal gas approxim
nd Paalanen (1981, 1982); ○ (curve 1): ^{3}He concentration $x_1 = 10^{-3}$ (
e 2): $x_2 = 3\times10^{-3}$ ($T_0 \sim 54$ mK), ■ (curve 3): $x_3 = 10^{-2}$ ($T_0 \sim 120$ m
, A_3 correspond to the values of $s_2^{(0)}(P; T = 0)/s_2(0; 0)$ in a field of $H =$
for the concentrations x_1, x_2, x_3.

Fig. 9

2. **Second sound velocity**. The second sound velocity – the sound velocity in a gas of ^{3}He quasi-particles – is determined by characteristic velocities of ^{3}He quasi- particles. Since these velocities change considerably in degenerate case (4.6), the second sound velocity s_2 strongly depends on polarization.

   The second sound velocity in polarized mixtures depends not only on polarization but also on frequency. One should separate between the high-frequency regime, $\omega\tau_d >> 1$ (τ_d is the dipole relaxation time), when the local densities of spins up and spin down, $N_{\uparrow(\downarrow)}$, do not change during the period of oscillations, and the low-frequency ("hydrodynamic") regime, $\omega\tau_d << 1$ when the particles are able to redistribute themselves between spin components adjusting $N_{\uparrow(\downarrow)}$ to the local (oscillating) values of thermodynamic variables. The velocity of the hydrodynamic second sound, $S_2^{(0)}$ (curve 2 in the figure) is smaller than the velocity S_2^{∞} of the HF second sound (curve 1).

3. The third major effect is associated with all sorts of polarization changes in ^{3}He$\uparrow$-^{4}He phase diagram. Again, these effects are much more pronounced at low temperatures when the ^{3}He chemical potentials $\mu_{\uparrow(\downarrow)}$ are equal to corresponding Fermi energies and, therefore, strongly depend on degree of polarization, (4.6).

   Here I will discuss only the polarization change in the limiting solubility of ^{3}He$\uparrow$ in ^{4}He when the mixture is in equilibrium with pure liquid ^{3}He$\uparrow$.

   The conditions of phase equilibrium correspond to equality of chemical potentials for up and down spins in a pure and dilute phases (cf. eq.(2.1)):

$$\mu_\uparrow^{(34)} = \mu_\uparrow^{(3)}, \mu_\downarrow^{(34)} = \mu_\downarrow^{(3)} \tag{4.14}$$

According to eq.(2.3), eq.(4.13) can be written at $T = 0$ as

$$\begin{aligned} \Delta' &+ \frac{\hbar^2}{2m_\uparrow^*}(6\pi^2 N_{3\uparrow})^{\frac{2}{3}} = \frac{\hbar^2}{2m_{3\uparrow}^*}(6\pi^2 N_{30\uparrow})^{\frac{2}{3}} \\ \Delta' &+ \frac{\hbar^2}{2m_\downarrow^*}(6\pi^2 N_{3\downarrow})^{\frac{2}{3}} = \frac{\hbar^2}{2m_{3\downarrow}^*}(6\pi^2 N_{30\downarrow})^{\frac{2}{3}} \end{aligned} \tag{4.15}$$

which coincide with (2.3) when $N_{3\uparrow} = N_{3\downarrow} = N_3/2$. To solve eqs.(4.14), one should know

- The total numbers of ^{3}He and ^{4}He particles in the system.
- The total magnetic moment (the total degree of polarization).
- The dependencies of effective masses of ^{3}He quasi-particles in the pure phase, $m_{3\uparrow(\downarrow)}^*$, and in the mixture $m_{\uparrow(\downarrow)}^*$, on polarization and concentration.

As a result, one will get the degrees of polarizations – *which are different in both phases, the polarization of dilute phase being considerably higher* – and the limiting ^{3}He concentration as a *function of the total polarization and the ratio of numbers of ^{3}He and ^{4}He particles in the system.*

To illustrate the magnitude of the effect, we will consider only the limiting case when

- the total number of ^{4}He particles is much smaller that the amount of ^{3}He, i.e. that the change in ^{3}He concentration in a droplet of ^{3}He$\uparrow$-^{4}He does not change the density and polarization of the ^{3}He$\uparrow$ bath.
- the effective mass of ^{3}He quasi-particles in the mixture is not very sensitive neither to ^{3}He concentration, nor polarization,

$$m_\uparrow^* = m_\downarrow^* = M \tag{4.16}$$

In this case, the numbers $N_{30\uparrow(\uparrow)}$ of spin-up and spin-down particles in the pure phase will be given by the total density of ^{3}He, N, and the degree of polarization, α

$$N_{30\uparrow(\downarrow)} = \frac{1}{2}N(1 \pm \alpha), \tag{4.17}$$

and eqs. (4.15) take the form

$$N_{3\uparrow(\downarrow)} = N\left[\frac{M}{m_{3\uparrow(\downarrow)}^*}\frac{(1\pm\alpha)^{\frac{2}{3}}}{2^{\frac{2}{3}}} - \frac{2M\Delta'}{(6\pi^2)^{\frac{2}{3}}\hbar^2 N^{\frac{2}{3}}}\right]^{\frac{3}{2}} \tag{4.18}$$

as far as the bracket is positive (otherwise, $N_{3\downarrow} = 0$). The resulting change in limiting ^{3}He concentration is

$$\frac{x_\uparrow}{x_0} = \left[\frac{(M/m_{3\uparrow}^*)(1+\alpha)^{\frac{2}{3}} + (M/m_{3\downarrow}^*)(1-\alpha)^{\frac{2}{3}} - 2M\Delta'/(3\pi^2)^{\frac{2}{3}}}{2(M/m_3^*) - 2M\Delta'/(3\pi^2)^{\frac{2}{3}}}\right]^{\frac{3}{2}} \tag{4.19}$$

A very rough estimate shows that the limiting concentration increases by about a factor 3 or 4 at complete polarization.

I want to repeat again that this result will change considerably if the amount of ^{4}He in the system would be comparable to or larger than the amount of ^{3}He.

In case of phase equilibrium between solid ^{4}He and liquid ^{3}He$\uparrow$-^{4}He mixture, the polarization leads to noticeable changes in both limiting concentration and pressures.

4. The last thermodynamic effect I want to mention is a rather peculiar temperature range at high polarizations in which the spin-up component of the mixture is degenerate,

$$T << T_\uparrow = \frac{\hbar^2}{2m_\uparrow^*}(6\pi^2 N_\uparrow)^{\frac{2}{3}},$$

while the spin-down component is so dilute, $N_\downarrow << N_\uparrow$, that it becomes "classical" and obtains the Boltzmann distribution function if

$$T_\downarrow = \frac{\hbar^2}{2m_\downarrow^*}(6\pi^2 N_\downarrow)^{\frac{2}{3}} << T$$

The properties of such a semi-degenerate system are different from both degenerate and Boltzmann solutions.

4.2 Transport and magnetokinetic effects

The most striking effects in transport are associated with already mentioned above effective suppression of s-wave interaction with polarization. This leads to an enormous growth of particles' mean free paths and transport coefficient.

The origin of the effect is quite simple. In case of s-wave scattering, the mean free path for spin-ups is restricted only by their collisions with spin-downs:

$$\ell_\uparrow \approx \frac{1}{N_\downarrow \sigma} \sim \frac{1}{N_\downarrow a^2} \tag{4.20}$$

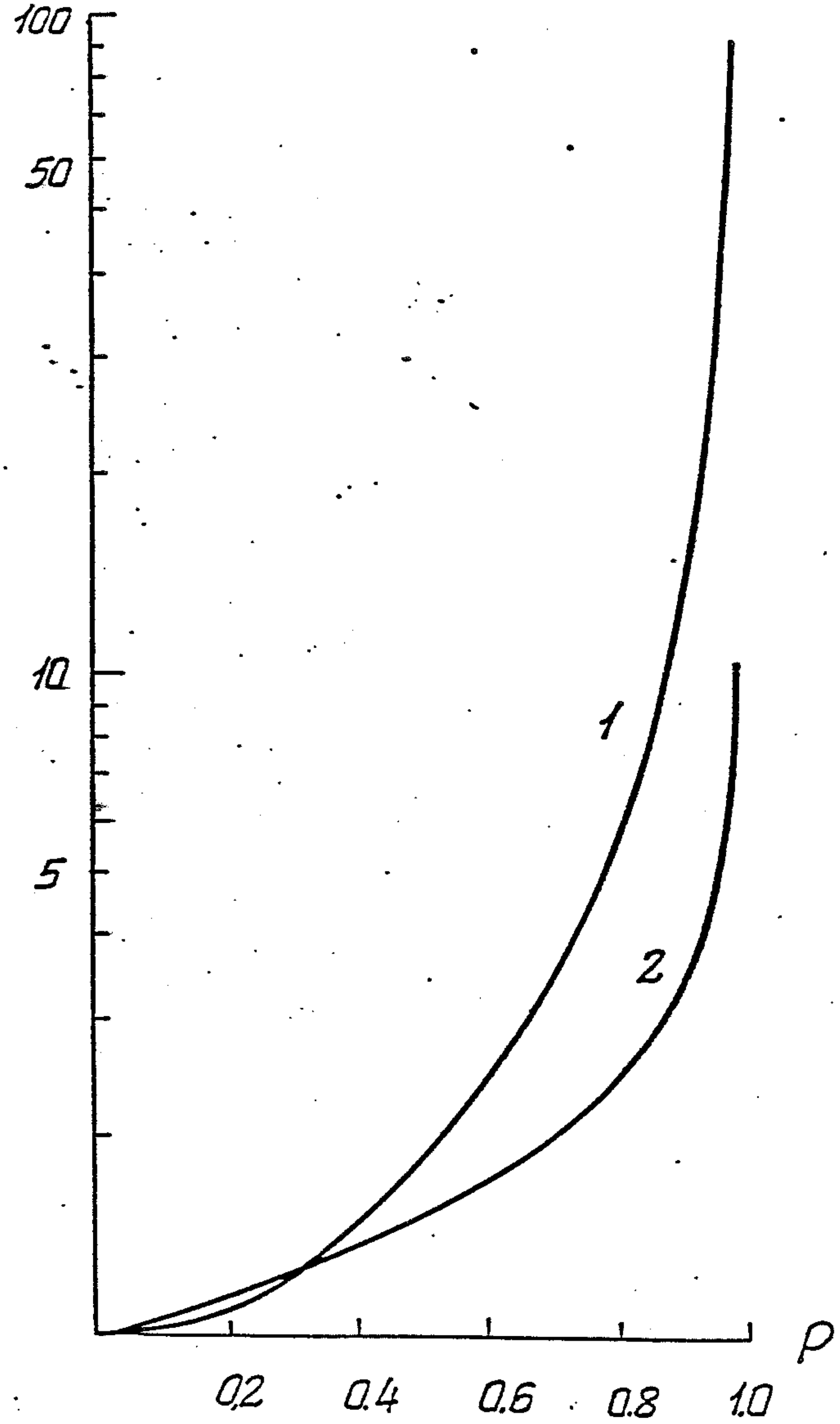
and increases practically without limit when the number of spin- downs goes to zero, $N_\downarrow \to 0$. Then $\ell_\uparrow$ is limited only by p-wave processes with much smaller cross-sections.

One can easily estimate the effective growth of mean free paths by comparing the cross-sections of s- and p-wave scatterings with amplitudes (1.24), (1.25)

$$\frac{\ell_\uparrow}{\ell_0} \sim \frac{\sigma_s}{\sigma_p} \sim \frac{a^2}{b^2 p^4} \sim (\frac{\hbar}{pa})^4 \sim \begin{cases} x^{-\frac{4}{3}} >> 1 & , \quad T << T_F \\ (\frac{E_0}{T})^2 >> 1 & , \quad T >> T_F \end{cases} \tag{4.21}$$

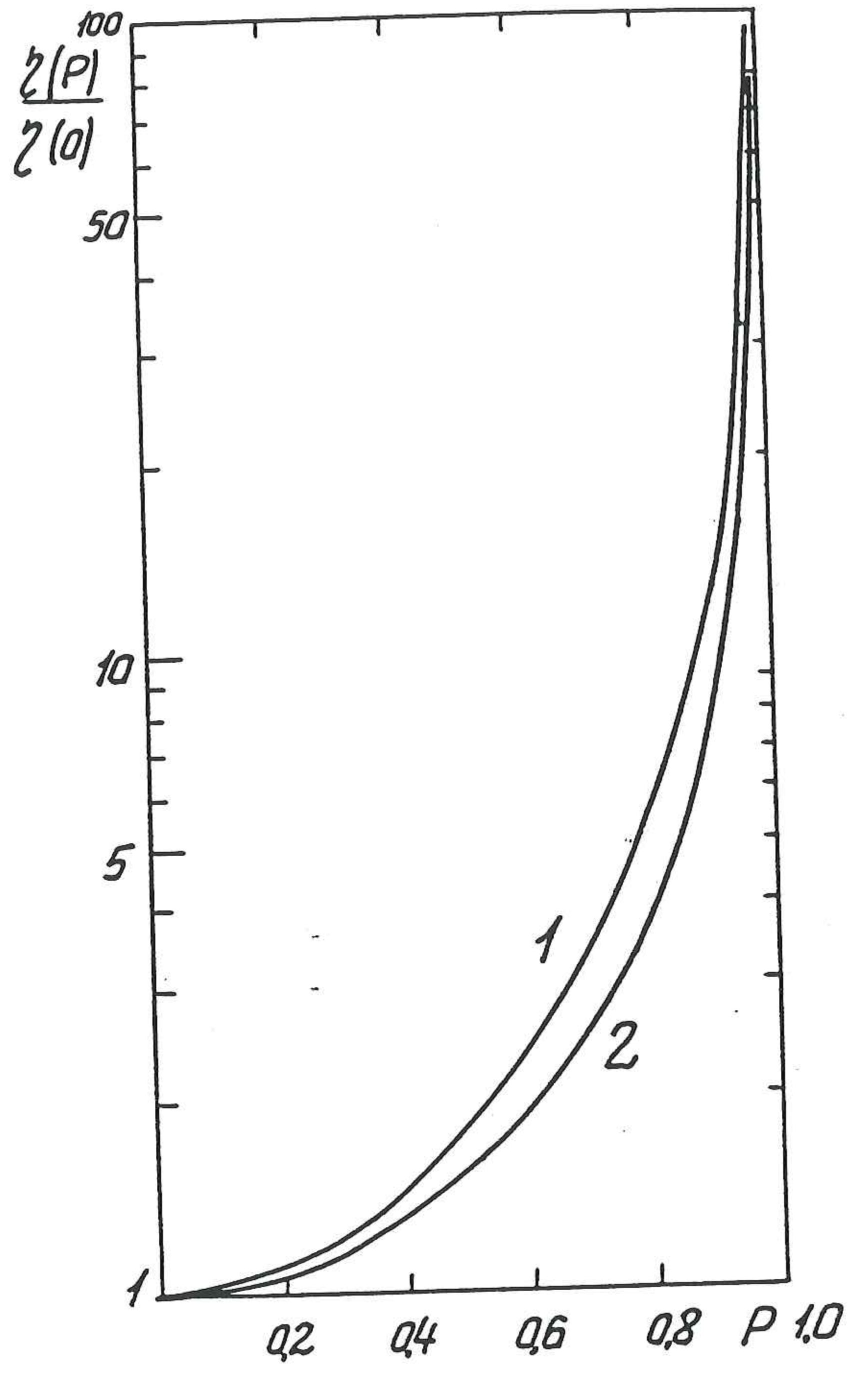
The enormous growth of mean free paths (4.20) is responsible for a corresponding growth of transport coefficients such as viscosity, η, and thermal conductivity, κ. The plots of relative growths of η (curve 1) and κ (curve 2) with polarization are shown in the figure for degenerate, $T << T_F$, case.

This effect is one of very striking macroscopic manifestations of the Pauli principle. Note, that this effect takes place irrespective of quantum degeneracy of the system, in both, "classical" Boltzmann and degenerate regimes. For the sake of comparison I have plotted in the next figure relative growths of viscosities $\eta/\eta(0)$ with polarization for degenerate (curve 1) and Boltzmann (curve 2) cases. Since these two curves are so close to each other, an estimate of change in viscosity in the intermediate temperature range is quite simple.



Polarization dependecies of relative viscosity (1) $\eta(P)/\eta(0)$ and thermal conductivity $\kappa(P)/\kappa(0)$ of

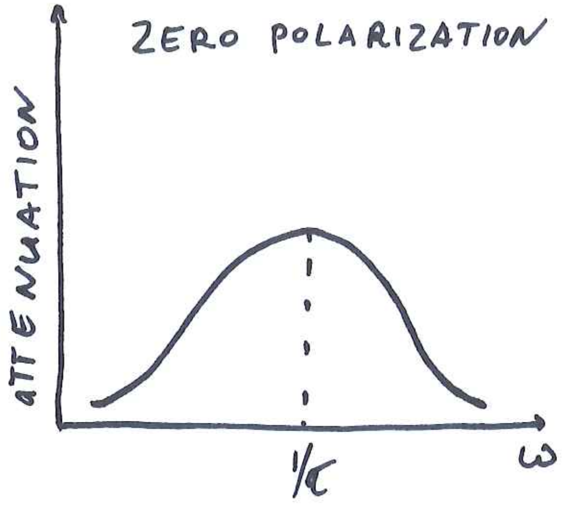
Fig. 10

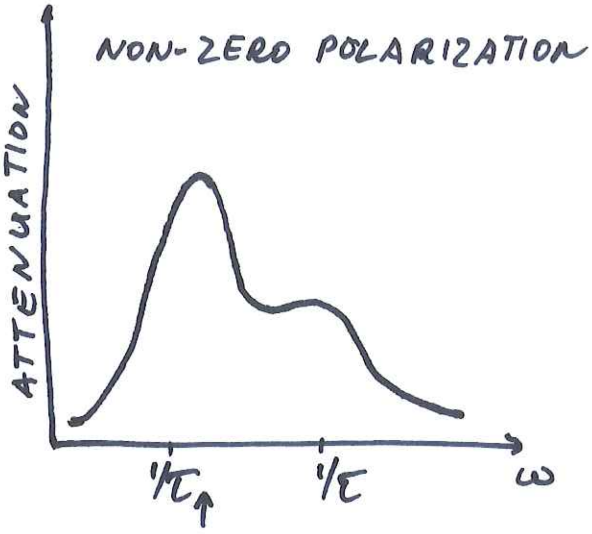


Polarization dependence of the relative change of the viscosity, $\eta(P)/$
1: degenerate solutions. $T \ll T_F$; 2: nondegenerate solutions, $T \gg T_F$

Fig. 11

Of course, an analogous effect should be observed also for a sound attenuation. However, the situation with the sound attenuation is even more interesting. Experiments on different frequencies may reveal a difference in relaxation times $\tau_{\uparrow(\downarrow)}$ for up and down spins. Schematically, the situation is presented in the figure:





at finite polarization one should observe a split of an attenuation peak at $\omega = 1/\tau$ (or at least a little bump) into two peaks one of which ($\omega = 1/\tau_\uparrow$) will increase in amplitude with polarization and shift towards lower and lower frequencies, while the other will stay at the same position but with decreasing amplitude.

The behaviour of spin diffusion coefficient is quite different from other transport coefficients. But before discussing diffusion, I would like to make a general comment concerning the transport phenomena in polarized systems.

If one does not change a direction of magnetization, then our system of spin-$\frac{1}{2}$ particles behaves exactly like a two-component mixture – a mixture of spin-up and spin-down components. Generally speaking, such *an analogy between a spin-polarized system and a multicomponent mixture is exact if the interaction between particles is of exchange origin.*

Therefore, thermodynamic and (longitudinal) transport phenomena in spin-polarized (dilute) ^{3}He$\uparrow$ systems can be described by the same expressions as corresponding phenomena in binary mixtures, while the degree of spin polarization

$$\alpha = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow) \tag{4.22}$$

plays the same role as the concentration of dissolved component in mixtures. If the interaction is an exchange one, then the total number of particles in each of the components does not change. Weak dipole processes with characteristic time τ_d for polarized systems are analogous to slow chemical reactions in case of binary mixtures.

In case of degenerate polarized systems, this analogy does not help much to calculate transport coefficients since the theory of mixtures of degenerate (Fermi) systems virtually does not exist. It is not so in case of Boltzmann systems where we can benefit from the relatively very well developed transport theory for multicomponent mixtures.

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Having said this, it would be quite natural to write the spin diffusion equation for polarized ^{3}He$\uparrow$ (mixtures) in the form standard for diffusion in binary mixtures. The (spin) diffusion current should contain the terms proportional to gradients of "concentration" (in our case – polarization), temperature and pressure (for ^{3}He$\uparrow$-He II mixtures – osmotic pressure)

$$\mathbf{J}_3 = -DN_3(\vec{\nabla}\alpha + \frac{\kappa_{st}}{T}\nabla T + \frac{\kappa_{sp}}{P}\nabla P) \tag{4.23}$$

where one could call $D\kappa_{st}/T$ and $D\kappa_{sp}/P$ the (spin-thermal diffusion and (spin) pressure diffusion coefficients.

The (longitudinal) spin diffusion coefficient D in (4.22), unlike viscosity and thermal conductivity, does not increase with polarization. On the contrary, in case of degenerate systems D goes to zero when the polarization $\alpha \longrightarrow 1$ reflecting the fact that at complete polarization not only the density of spin-down particles $N_\downarrow \longrightarrow 0$, but their velocity, $v_\downarrow \propto N_\downarrow^{\frac{1}{3}}$, goes to zero too (the condition of spin diffusion corresponds to the lack of total mass current meaning the exact balance between opposite currents of spin-ups and spin-downs). In the Boltzmann, $T >> T_F$, case D practically does not depend on polarization reflecting the well-known fact of independence of diffusion coefficient in a mixture of ideal gases of the mixture concentrations.

I do not have enough time to discuss here the properties of spin thermal diffusion and pressure diffusion ratios, κ_{st} and κ_{sp}; I would only like to mention that the theory of spin pressure diffusion is very interesting and not at all trivial even in simplest cases.

4.3 Spin dynamics and spin waves

Let us go back and write in more detail accurate macroscopic ("hydrodynamic") equations of spin dynamics.

In exchange systems spin is conserved,

$$\frac{\partial \mathbf{M}}{\partial t} + \frac{\partial}{\partial x_i} J_i = 0, \tag{4.24}$$

while the spin current $\mathbf{J}_i$ ($\mathbf{J}_i$ is a vector in both, spin and usual spaces) should be proportional to the spatial gradients of thermodynamic variables:

$$J_i^\alpha = M^\alpha v_i + A_T^\alpha \nabla_i T + A_p^\alpha \nabla_i P + D^{\alpha\beta}\nabla_i M^\beta \tag{4.25}$$

Since *there is only one preferred direction in the spin space, namely the direction of magnetization,*

$$\mathbf{m} = \mathbf{M}/M, \tag{4.26}$$

the most general form of coefficients in eq.(4.24) is

$$\mathbf{A}_T = \mathbf{m}A_T, \mathbf{A}_p = \mathbf{m}A_p \tag{4.27}$$

$$D^{\alpha\beta} = D_1\delta_{\alpha\beta} + D_2 m_\alpha m_\beta + D_3 e_{\alpha\beta\gamma} m_\gamma \tag{4.28}$$

Then, if one slightly changes notations,

$$A_T = -\beta N_3 D\kappa_{st}/T, A_p = -\beta N_3\kappa_{sp}/P, \tag{4.29}$$

$$D_1 + D_2 = D, D_3/D_1 = \Omega\tau_\perp, D_1(1 + D_3^2/D_1^2) = D_\perp, \tag{4.30}$$

$$2\beta H = \hbar\Omega_0, \tag{4.31}$$

the equations of spin dynamics (4.23), (4.24) take the form of the following equation in the vector of spin polarization $\vec{\alpha} = \mathbf{M}/\beta N_3$:

$$\begin{aligned} & N_3(\dot{\vec{\alpha}} + v_\kappa\nabla_\kappa\vec{\alpha}) + \Omega_0[\mathbf{m}\times\vec{\alpha}] - \\ & \quad -\nabla_\kappa \Bigg[\frac{D+N_3}{1+\Omega^2\tau_\perp^2}\left(\nabla_\kappa\vec{\alpha} + \Omega\epsilon_\perp[\mathbf{m}\times\nabla_\kappa\vec{\alpha}]\right. \\ & \quad \left. +\Big(\frac{D}{D_\perp}\Big(1+\Omega^2\tau_\perp^2\Big) - 1\Big)\,\mathbf{m}(\mathbf{m}\nabla_\kappa\vec{\alpha})\right) - \\ & \quad -DN_3\mathbf{m}\Big(\frac{\kappa_{st}}{T}\nabla_\kappa T + \frac{\kappa_{sp}}{P}\nabla P\Big)\Bigg] = 0 \end{aligned} \tag{4.32}$$

Though eq.(4.31) looks rather cumbersome, in reality it is quite simple. Its longitudinal component (component along $\mathbf{m}$) is exactly eq.(4.22) and does not interest us here any more. The transverse component of eq.(4.31) (component perpendicular to magnetization $\mathbf{m}$) can be rewritten as a following equation in the circular component M_+ of magnetization $\mathbf{M}$, $M_+ = M_x + iM_y$:

$$\frac{\partial}{\partial t}M_+ + i\Omega_0 M_+ - \frac{D_\perp}{1+\Omega^2\tau_\perp^2}(1 + i\Omega\tau_\perp)\nabla^2 M_+ = 0 \tag{4.33}$$

Equation (4.32) describes the inhomogeneous rotation of magnetization (spin waves) and is somewhat analogous to the well known Leggett(-Rice) equation of spin dynamics in Leggett's notation $\Omega\tau \to \mu M$. However, we still have several not very clear points:

- What is Ω?
- Are the spin diffusion coefficients $D_\perp$ and D equal or not?
- What is $\tau_\perp$?
- What is the value of $\Omega\tau_\perp$?

Before answering these questions, let us write the solution of eq.(4.32) in case of plane waves,

$$M_+ = M_{+0}\exp(-i\omega t + i\mathbf{vr}),$$

i.e. *the spectrum of spin waves:*

$$\omega = \Omega_0 + \frac{D_\perp}{\Omega\tau_\perp}\frac{1}{1+1/\Omega^2\tau_\perp^2}(1 - i/\Omega\tau_\perp)^2_\kappa \tag{4.34}$$

Now back to the above questions:

1. Ω is the frequency of spin oscillations in the *molecular field.* In dilute low-temperature systems in the main approximation

   $$\Omega \approx -\frac{4\pi a\hbar}{M}(N_\uparrow - N_\downarrow) \tag{4.35}$$

2. The relaxation time approximation for collision spin current ($\hat{L}(\hat{n})$ is the collision operator) should contain two different relaxation times, $\tau_\perp$ and τ for the system with preferred direction:

   $$Tr_\sigma \int d\Gamma v_i \vec{\sigma}\hat{L}(\hat{n}) = \frac{1}{\tau_\perp}\mathbf{J}_i - (\frac{1}{\tau} - \frac{1}{\tau_\perp})\mathbf{m}(\mathbf{m}\mathbf{J}_i) \tag{4.36}$$

   The transverse and longitudinal spin diffusion coefficients, $D_\perp$ and D, are proportional to the corresponding relaxation times, $\tau_\perp$ and τ:

   $$D = \tau N_3 \left[N_\uparrow < v^{-2} >_\downarrow + N_\downarrow < v^{-2} >_\uparrow\right] \tag{4.37}$$

   $$D_\perp = \frac{\tau_\perp}{3}\frac{N_\uparrow < v^2 >_\uparrow - N_\downarrow < v^2 >_\downarrow}{N_\uparrow - N_\downarrow} \tag{4.38}$$

   where $< \cdots >_{\uparrow(\downarrow)}$ is the average of the corresponding velocity over the distribution function for spin- ups (downs).

   It happens, that in Boltzmann region $\tau \approx \tau_\perp$, $D \approx D_\perp$. However, in degenerate case $\tau > \tau_\perp$, while at high polarizations $D_\perp > D$. What is very important, is that at low temperatures and high polarizations $\tau_\perp^{-1}$, unlike τ^{-1}, contains a contribution without a small factor $(T/T_F)^2$.

3. $\Omega\tau_\perp$ is the quality factor of the spin waves and characterizes the attenuation. At low temperatures/concentrations,

   $$\Omega\tau_\perp \sim \alpha\frac{\Lambda}{a},$$

   and the spin waves experience a very small attenuation as soon as the de Broglie wavelength of particles Λ exceeds a.

   The figure compares two sets of recent experimental data on $D_\perp/\Omega\tau_\perp$ with calculations in the simplest s-wave approximation.

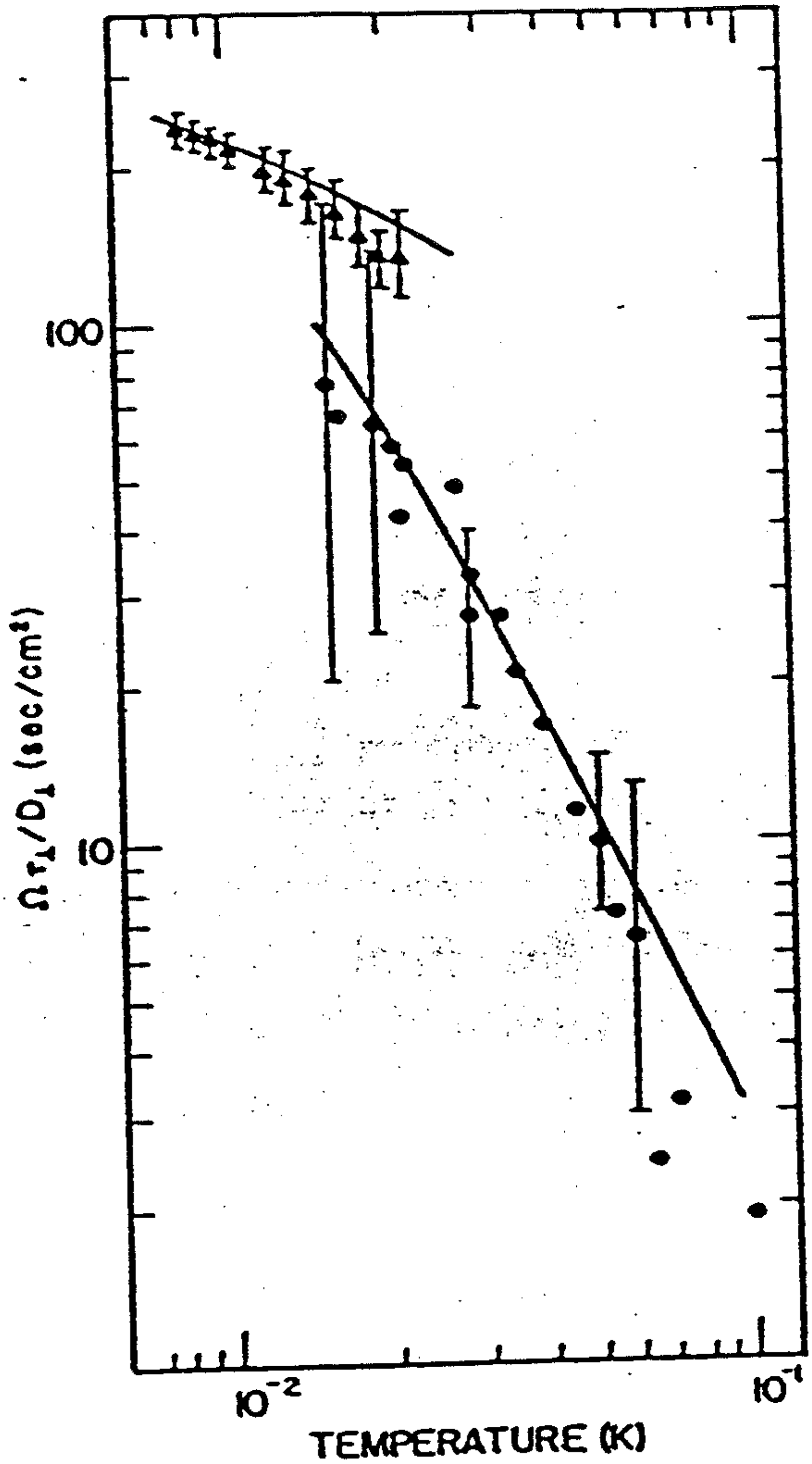


Fig. The temperature dependence of the ratio of the quality factor for spin waves and the transverse spin-diffusion coefficient, $\Omega\tau_\perp/D_\perp$. The ▲ indicate the experimental data and calculations with the s-wave scattering length $a \simeq -0.5$ Å (concentration $x = 3.7 \times 10^{-4}$, $T_F \simeq 13$ mK, $H =$ 89 kOe); and the ● indicate data with analogous calculations with $a \simeq 0.77$ Å ($x = 1.82 \times 10^{-3}$, $T_F \simeq 39.8$ mK, $H =$

5 Dense ^{3}He$\uparrow$ liquid systems

5.1 Methods of polarizations

Different methods of polarization for pure liquid ^{3}He$\uparrow$ and concentrated ^{3}He$\uparrow$-^{4}He liquid mixtures have already been described above in chapter 4.1. I will only list them again:

- **Brute force technique.**
  Advantages: Thermodynamically equilibrium polarized state at ultra-low temperatures.
  Disadvantages: Very low degree of spin polarization.

- **Melting of polarized solid ^{3}He$\uparrow$.**
  Advantages: High degree of polarization.
  Disadvantages: Finite (often not very long) lifetime; strong heating; non-stationary and non-equilibrium conditions; not very clear dynamic of melting; high field (difficulties for NMR).

- **Condensation of polarized gas ^{3}He$\uparrow$.**
  Advantages: Relatively long lifetime; high degree of polarization; low field (simple NMR measurements).
  Disadvantages: Non-stationary and non-equilibrium state; small amount of ^{3}He; semi-2D character of the system (thick film).

- **HF methods.**
  Advantages: Stable degree of polarization; technically not overcomplicated method.
  Disadvantages: Low polarizations; unclear (but important) coupling to the substrate; possible heating.

- **use of superleaks to take away ^{4}He.**

5.2 Spin-polarized Fermi liquids

Dense pure liquid ^{3}He$\uparrow$ and concentrated ^{3}He$\uparrow$-^{4}He liquid mixtures are two ^{3}He$\uparrow$ Fermi liquids with different parameters.

It is impossible to calculate accurately the parameters of these systems microscopically starting from the first principles. The difficulties are related not to the polarization of the spin system, but with the lack of non-model theories for dense systems. What is even worse, it is very often impossible to prove unambiguously the correctness of practically any model and to estimate the degree of its accuracy. The reason is quite simple: helium systems have no intrinsic small parameters, nearly all dimensionless combinations of physical characteristics are of the order of 1, and therefore, the model predictions reflecting correctly different features of helium systems, do not have parameterical accuracy.

I will illustrate this on the example of, say, viscosity. Obviously, the changes in viscosity at small polarizations are quadratic in polarization α,

$$\delta\eta(\alpha)/\eta(0) = \gamma\alpha^2 \tag{5.1}$$

Different models for ^{3}He (glass-type, localized, metamagnetic, spin-fluctuations, etc.) predict reasonable – of the order of 1 – values of γ. Therefore, since the accuracy of these models is unknown and all of them are basically reasonable emphasizing different features of ^{3}He, it would not be very wise to segregate between them choosing as the "best" one the one which "predicts" the value of γ (5.1) closer to the (current) experimental value $2 \div 4$ than the other ones. Such a segregation would be possible only if predictions and experiment would differ by orders of magnitude or have distinctively different singularities. By the same token, it would be in same sense a waste of time to try to modify some model description in order to have better and better quantitative agreement with (current) experimental data. Such a modification is practically always possible. It never proves anything, but just makes an underlying model less general and attractive, and more vulnerable to criticism.

Below I will base the brief discussion of polarization characteristics of ^{3}He$\uparrow$ Fermi liquids on the comparison with the features of dilute ^{3}He$\uparrow$ systems discussed in previous four sections.

Some of the common features of dense and dilute ^{3}He$\uparrow$ systems are due to the exchange origin of ^{3}He interaction. Therefore,

- In dense ^{3}He$\uparrow$, like the dilute phases, longitudinal and transverse phenomena are separate from each other.

- In describing the longitudinal effects (thermodynamics and transport), one may use an analogy with a multicomponent (binary) Fermi liquid.

- All thermodynamic characteristics can be expressed via harmonics of the Fermi liquid function using the standard rules. The spin structure of the Fermi liquid function is the same as (4.11):

$$\begin{aligned} F_{\sigma\sigma'}(\mathbf{p},\mathbf{p}') &= F_1(\mathbf{p},\mathbf{p}') + F_2(\mathbf{p},\mathbf{p}')\vec{\sigma}\vec{\sigma}\,' \\ &+ [F_3(\mathbf{p},\mathbf{p}')\vec{\sigma} + F_3(\mathbf{p}',\mathbf{p})\sigma']\,\mathbf{m} + \\ &+ F_4(\mathbf{p},\mathbf{p}')(\mathbf{m}\vec{\sigma})(\mathbf{m}\vec{\sigma}\,') \end{aligned} \tag{5.2}$$

- Transport coefficients are very complicated functions of the Fermi liquid harmonics and scattering probabilities for quasi-particles.

Some of the differences are rather quantitative than qualitative.

- The polarization of spin system suppresses the s-wave scattering in dense systems too. But, since
  - the F-function is not the scattering amplitude (vertex function) and is only related to it via some integral equation.

- the angle between momenta $\mathbf{p}, \mathbf{p}'$ of quasi- particles in the F-function $F(\mathbf{p}, \mathbf{p}')$ is not the same as the scattering angle in the vertex,

the suppression of the s-wave scattering does not mean a disappearance of any harmonics in the F-function. However,

- One usually assumes that the harmonics of the vertex decrease with their number. Therefore, the disappearance of the main (zeroth) harmonic in the exact vertex (not F-function!) should lead to some decrease in effective interaction.

All this means that the polarization leads to two competing effects. One is an increase in density of states (here the effect of polarization is similar, but not equal to the effect of pressure), thus tending to increase the effective mass, sound velocity, etc. However, some decrease in effective interaction related to the disappearance of the s-wave scattering from the bare vertex, pushes the system in the opposite direction towards a decrease in effective mass, sound velocity, etc., and towards increase in transport coefficients. The second tendency is visible at least in current measurements of viscosity (γ eq. (5.1) is positive).

5.3 Spin dynamics and spin waves

At first glance, the macroscopic equations of motion (4.23)- (4.32) are quite general, and the only thing one has to do is to express the molecular field Ω through the Fermi liquid harmonics, and the relaxation time $\tau_\perp$ through the scattering probabilities and F-function. Then one ends up with a Silin-Leggett description of spin waves in spin-polarized Fermi liquid.

A detailed analysis shows that such a procedure is absolutely legitimate if one deals with a dilute system (like dilute ^{3}He$\uparrow$-^{4}He mixtures) at arbitrary polarizations or dense Fermi liquid at low polarizations.

However, if the system is dense <u>and</u> the polarizations are high, the underlying procedure becomes very problematic. I will list several of the problems. Some of them are already understood, but some are still very unclear.

- The transverse relaxation time $\tau_\perp$ does not necessarily contain a large factor $T_F/T)^2$. Therefore, $\tau_\perp$ is not always large, and the applicability of the single-particle (Landau) kinetic equation becomes questionable. Without the kinetic equation, the (semi-) microscopic description may become ungrounded.

- One cannot simply ignore contributions to the spin current (4.24)

$$D^{\alpha\beta\gamma}\nabla_i M^{\beta\gamma} \tag{5.3}$$

  which are quadratic in magnetization/density.

- The non-local corrections to the Fermi liquid interaction are not negligible any more. Here one should write the main equation of the Fermi liquid theory in the form

$$\begin{aligned}\delta\epsilon_\sigma(\mathbf{p}) = &\int F_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')\delta n_{sigma'}(\mathbf{p}')d\Gamma' \\ &+ \int F^{(1)}_{i\sigma\sigma'}(\mathbf{p},\mathbf{p}')\nabla_i \delta n_{\sigma'}(p')d\Gamma' + \\ &\int F^{(2)}_{i\kappa\sigma\sigma'}(\mathbf{p},\mathbf{p}')\nabla_i\nabla_\kappa \delta n_{\sigma'}(p')d\Gamma' + \cdots\end{aligned} \tag{5.4}$$

while F is the only function entering the standard theory.

- Though new Fermi liquid functions $F_i^{(1)}$, $F_{i\kappa}^{(2)}$ in eq.(5.4) are also expressed through the exact vertex part and its derivatives, some of their components originate from the so-called off-shell terms and are complex. This leads to a noticeable attenuation of spin waves.

In summary, one should conclude, that

- *spin waves in dense and highly polarized Fermi liquid ^{3}He are described by the equations very different from those for either dilute systems, or systems with low polarizations.*
- *spin waves in dense and highly polarized Fermi liquids are accompanied by a high attenuation.*

One needs an experiment to make a positive conclusion whether the spin waves can propagate in these conditions or not!

6 Conclusions. Experimental possibilities

At present we have a pretty reasonable quantitative understanding of most of the phenomena in dilute polarized ^{3}He$\uparrow$ systems and a more qualitative description for dense systems. As I see it, we have the following main problems for theory of

- boundary effects including slip effects and boundary depolarization.
- superfluid transition temperature for ^{3}He in ^{3}He-^{4}He mixtures.
- spin waves in dense polarized Fermi liquids.

These problems, especially the last two, are very fundamental, difficult, and important.

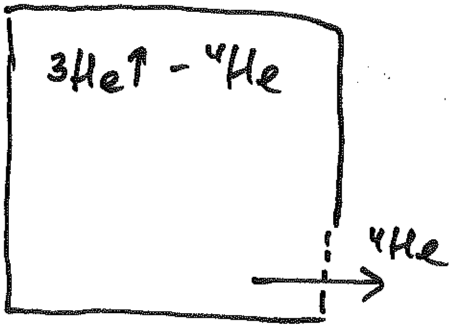
I would also like to list some of the experiments for ^{3}He- ^{4}He mixtures which I fill to be very interesting and enlightening (see also the end of section 2).

1. Accurate experiments at higher pressure.
2. Osmotic pressure.
3. Second sound velocity.

4. Sound attenuation.
5. Viscosity.
6. Spin diffusion.
7. Magnetic susceptibility.
8. Systematic study of T_1 (in the bulk and at the walls).

   These 8 experiments (described in more detail in section 2) have nothing to do with spin polarization, but seem to be good sources of information on the parameters of the mixtures.

9. (Non-equilibrium) polarization of ^{3}He with a superleak



10. Possible spontaneous penetration of ^{3}He atoms into solid ^{4}He after a threshold polarization of the melt around 30% at low temperatures.
11. Observation of large changes in the phase diagram with polarization.
12. Splitting of the sound attenuation peak in $^3\mathrm{He}\uparrow$-^{4}He at high frequencies.
13. Polarization dependences of the osmotic pressure and second sound velocity.
14. Systematic study of magnetokinetic effects (growth of transport coefficient with polarization).
15. Observation of anisotropy of spin diffusion.
16. Other NMR (spin echo and spin wave) experiments.

   And the most exciting two:

17. Possibility of spin waves propagation in dense Fermi liquids.
18. Observation of ^{3}He superfluidity in ^{3}He-^{4}He mixtures with and without polarization.

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